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PATTERNS OF THINKING IN SOLVING PROBLEMS

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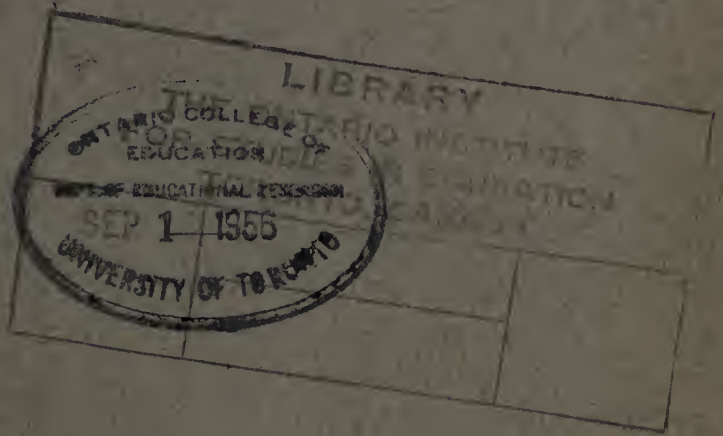
GUY T. BUSWELL

With the coöperation of

BERT Y. KERSH

UNIVERSITY OF CALIFORNIA PUBLICATIONS IN EDUCATION

Volume 12, No. 2, pp. 63-148, 16 figures in text



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FOREWORD

THE STUDY reported in this monograph is concerned with the mental operations of a group of subjects *during the process* of solving a problem. The method is primarily observational and analytical. The aim is to obtain evidence of individual patterns of thinking of a kind that will permit objective descriptions. The study is not experimental; it is not a comparison of controlled groups; it does not assume that the sample of subjects studied is typical of the general population. The purposes are, first, to develop a technique by which patterns of thinking may be compared objectively, and second, to add to the sum of knowledge regarding the nature of the problem-solving process. It is hoped that the findings will be of value to the schools in their attempt to "teach students how to think."

WILLIAM A. BROWNELL

Dean of the School of Education

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PATTERNS OF THINKING IN SOLVING PROBLEMS

BY

GUY T. BUSWELL

CHAPTER I. INTRODUCTION

TWO CONTRASTING treatments of problem solving have had a marked effect on school practices. One of these is represented by Dewey's well-known group of categories or steps under which the various aspects of the process of problem solving may be grouped. These steps represent his abstracted ideal of the thinking process at its best. It is worth noting that Dewey's treatment resulted from a logical analysis of the act of thinking rather than from empirical research on the sequential thinking of actual people. Another body of literature on problem solving, represented by such writers as Wertheimer, Duncker, and Lazerte, is based on an analysis of the process of problem solving as revealed in descriptions of problem-solving thinking by different individuals. These down-to-earth studies have the special merit of revealing both the errors and the successes in thinking, and they are of greater potential value as aids to teaching and learning than the more polished, ideal descriptions of logical steps in solving problems.

PROBLEM

This report¹ is concerned with factors that enter into patterns of problem-solving thinking as revealed through tests and recordings of high school pupils and university students as they attempted to solve six sets of problems. The problems were expressed in arithmetical terms, but the study is concerned with processes of thinking rather than with specific arithmetical outcomes.

The purpose of the study is to find whether there are general patterns of problem-solving thinking which characterize sizable groups of individuals, or whether the thinking process is so varied in character that only descriptions of individual thinking can be given. It is worth noting that the studies of the thinking process so far published have made little headway in describing patterns of thinking.

Two general levels of operation will be identified in the process of problem solving. First, in what might be described as an external approach to the problem, the following factors will be studied: how

¹ This study was supported by grants from the Research Fund of the Department of Education, University of California, Berkeley.

the problem is read, how the data in the problem are recognized, and whether the subject begins by estimating what the answer will be. To provide evidence on such external factors, group tests were used. A second level of operation was concerned with the details of the thinking process during the actual solving of a problem. Here the method was necessarily that of individual analysis, the main objective being to identify the sequence of mental operations as the subject tries to find a solution to the problem. In this part of the research the main difficulty is to obtain objective evidence of the subjective thinking process. Furthermore, the categories or patterns of thinking employed must grow out of the thinking processes of the subjects rather than be formulated in advance and the subject's mental operations then fitted into such a prestructured scheme. The major effort is first to find and describe the sequences of thinking and then to see whether these sequences fall into any general patterns. Pattern, as used here, means any consistent sequence of operations traced through a complex of possible operations available to the problem solver.

One further characteristic of the study should be made clear at the outset. The methods employed are primarily descriptive rather than experimental. The major interest is in finding qualitative differences in the process of problem-solving thinking. For example, when individuals of a certain group are faced with the same problem, what kinds of mental operations do they employ in attempting to solve it? First, how large a catalogue of operations or processes can be identified and possibly categorized? Second, when a given individual is placed against a background of possible operations, what is the sequence and pattern of his thinking? To what extent can common patterns be found? There is no presumption that the sample of cases studied is typical of any larger sample or that prediction of problem-solving behavior can be made for other populations not studied. The central interest is in how people think when attempting to solve problems, as revealed by the study of 499 high school and college students. If the study is successful in developing a more objective technique for identifying characteristics of the thinking process, it may make possible a more reliable procedure for gathering data which may then be treated quantitatively by more rigid statistical procedures.

RELATED STUDIES

No attempt will be made in this report to review the literature on problem solving. However, five studies which are somewhat directly related to the purposes of the present investigation should be mentioned.

In 1933, Lazerte² reported a study of the solving of arithmetical problems, a part of which has a direct bearing on the method of analysis used here. The "Envelope Test" reported in chapter two of Lazerte's monograph was designed to produce objective evidence of the processes employed by individuals in solving arithmetical problems. Lazerte structured the problems into a series of steps of solution so that two alternative procedures, typed on two envelopes, were offered to the subject after he had read the problem. Within each of the two envelopes were two additional envelopes, each offering a choice of two further alternatives for solving the problem. Similarly, a third pair of solutions was offered for each alternative that might be chosen. In the final stage, eight possible alternatives were available for choice. Lazerte was thus able to describe objectively the process of thinking followed by the problem solver. Although Lazerte's technique was restricted to alternatives based on arithmetical logic, it did point the way to securing analytical data on problem solving by objective means.

In 1935, Duncker published in Berlin an important study, *On Problem-Solving* (translated into English in 1945).³ In his investigations he carried the analysis of the act of thinking much further than Lazerte but did not use the objective controls that Lazerte had used. Although Duncker's report is a major contribution to the qualitative analysis of thinking, he does not go so far as Lazerte in showing quantitative patterns. However, his graphic analysis of logical alternatives in problem-solving thinking (p. 5) might have been made the basis for a quantitative study of patterns of thinking had it been applied to a sufficiently large group of subjects.

Wertheimer's study, *Productive Thinking*, was published in 1945.⁴ This, like Duncker's monograph, is a study of mental processes employed in thinking and is notable for its success in penetrating below surface observations. The concrete case accounts reported are rich in detail regarding the nature of the thinking process but give no clue to whether—or to what degree—the kinds of thinking described are representative. Few, if any, studies have been as successful in reaching the "grass roots" of the thinking process.

An attempt to introduce some further control into the study of the

² M. E. Lazerte, *The Development of Problem-solving Ability in Arithmetic*. Toronto: Clarke, Irwin and Company, 1933, pp. 5-21.

³ Karl Duncker, *On Problem-Solving*. Washington: American Psychological Association, 1945. ix + 113 pp. (Translated by Lynne S. Lees.)

⁴ Max Wertheimer, *Productive Thinking*. New York: Harper and Brothers, 1945. xi + 224 pp.

thinking process was reported in a study by Bloom and Broder⁶ in 1950. By combining the multiple-choice technique of standardized tests with the method of "thinking out loud" they were able to describe more objectively the procedures employed in thinking. Their study was carried out with college students, but the method would be applicable to students at lower levels. The descriptions of thinking processes are carefully recorded in this monograph, and in the appendix a check list is provided which organizes the characteristics of problem solving, but again without quantitative data to distinguish patterns of thinking from individual variations.

After our study had been under way for a year, a related line of research which was being carried on at the University of Illinois under the direction of Francis G. Cornell was reported by Dora Damrin⁶ at the annual meeting of the American Educational Research Association, in February, 1953. The use of the "Tab Item" technique there reported has been extended in a doctor's dissertation by Joseph L. Saupe.⁷ These studies have contributed to an objective method of analyzing problem-solving thinking. Although the data are not organized to show patterns of thinking, they obviously could be so organized, and they contribute significantly to this field of research.

METHOD AND PROCEDURE

The data for this study were obtained from 209 high school pupils and 290 college students—499 subjects in all. The data consist of scores on tests constructed for use in this study, and of descriptions, or protocols, based on voice recordings made as the subjects solved some of the problems. The time taken for the different parts of the test on generalizations was recorded by an electrically driven tape kymograph. The tests were given in the years 1952 and 1953. The average time required for the group tests was forty-five minutes. A full fifty-minute period was allotted for individual tests, although the actual time used by the subjects varied. The high school subjects were obtained from two large high schools in Oakland and Alameda, California, the college students from the University of California in Berkeley. A summary of the sub-

⁶ Benjamin S. Bloom and Lois J. Broder, *Problem-solving Processes of College Students*. University of Chicago Press, 1950. 109 pp. (Supplementary Educational Monograph, No. 73.)

⁶ Robert Glaser, Dora E. Damrin, and Floyd M. Gardner, *The Tab Item: A Technique for Measurement of Proficiency in Diagnostic Problem Solving Tasks*. Urbana: University of Illinois, Bureau of Research and Service, College of Education, 1952. 13 pp.

⁷ Joseph L. Saupe, *Trouble Shooting Electronic Equipment*. Urbana: University of Illinois, Bureau of Educational Research, College of Education, 1954. 127 pp.

jects used and the data available from each group is given in table 1.

The tests used were constructed so as to reveal various characteristics of problem solving. The purpose was to identify qualitative differences rather than to set standards of achievement on the test or to make

TABLE 1
SUBJECTS AND AVAILABLE DATA

Subjects			Available data					
Source	Group	N	Test scores				Recordings of protocols	Card analyses
			Tests				Tests	Test
			1	2	3	4	5 6	6
College University of California	Fall, 1952	108	X	X	X	X	X X	..
	Spring, 1953	89	X	X	X	X	.. X(15)	X(15)
	Fall, 1953	93 (290)	X	X	X	X	.. X	X
High School Oakland..... Alameda.....	Summer, 1953	108	X	..	X	..	.. X	X
	Fall, 1953	101 (209)	X	X	X	X		..
Total high school and college...		499						

inferences of group behavior from samples employed. The six parts of the test were designed to help answer the following questions:

1. In solving a problem, to what extent do students first structure their thinking in terms of relevant, irrelevant, or needed facts?
2. Can students estimate what would be a reasonable answer before proceeding to the solution of a problem? How are such estimates made?
3. How far is the pattern of thinking affected by the verbal sequence of the different parts of the problem?
4. Can students think more effectively when letter symbols are used than when quantities are stated in numbers?
5. How adept are students in discovering a generalization which covers a group of similar problem situations? What processes of thinking are used in making generalizations?
6. Can patterns of problem-solving thinking be identified and described? Can a method be devised for describing patterns more objectively?

The report of the study will be organized as follows: Chapter ii will deal with the four group tests and will provide data on the first four of the preceding questions. These data are preliminary to the main problems of the study and provide only an external view of the process

of problem solving. Chapter iii will report a study of the thinking processes involved in discovering a generalization. The principal data consist of protocols secured through voice recordings of the subjects while "thinking aloud" and of written answers to the exercises that made up the test. Chapters iv and v will report the central part of the study. The purpose is to find whether there are general patterns of problem solving which characterize sizable groups of subjects, and if such patterns are found, to study the processes of thinking of individual subjects in the light of whatever group patterns may be discovered. In chapter vi the results of the study will be discussed in reference to methods of teaching and learning.

CHAPTER II. GROUP TESTS OF PROBLEM SOLVING

To indicate something of the background with which students attempt the solution of problems, a set of four tests was constructed. A copy of the tests, in reduced type, is shown in the Appendix. All the subjects in the study took the four tests, although the 108 in the Oakland group were given only parts of the tests, and these in modified form, as will be explained later. The tests dealt respectively with (1) selection of data, (2) estimation of answers, (3) problem-solving sequence, and (4) symbolic thinking. They should be considered as preliminary to the main part of the study, which deals with individual analyses of the problem-solving process. The purpose of the four tests is to give a general view of the way problems are approached. There has been no attempt to scale the problems according to difficulty. The results will be presented, test by test, in the remainder of this chapter.

TEST 1, SELECTING DATA

Although many of the problems encountered in school are expressed in terms so explicit that the data given fit exactly the requirements for their solution, some problems encountered in school and most problems in life are not expressed so neatly. Rather, a common characteristic of problems is that irrelevant facts are mixed in with those that are pertinent, and at times insufficient facts are supplied for solution of the problems. Test 1 is made up of problems of this kind, the person taking the test being required to indicate which of the facts given are relevant and which are irrelevant and, if some facts are missing, what additional data are needed to make the problem solvable. One of the problems, for example, was this: "On January 14, Mr. Smith and his son, who was 17 years old, drove to market with some sheep which they sold

TABLE 2
IDENTIFICATION OF RELEVANT, IRRELEVANT,
AND NEEDED FACTS

Score on Test 1	Frequency		
	High school (Alameda)	College	Total
22.....	..	..	..
21.....	2	1	3
20.....	3	4	7
19.....	2	9	11
18.....	4	8	12
17.....	3	19	22
16.....	8	11	19
15.....	8	22	30
14.....	14	28	42
13.....	7	44	51
12.....	13	47	60
11.....	11	44	55
10.....	10	20	30
9.....	7	13	20
8.....	5	2	7
7.....	2	1	3
6.....	1	2	3
5.....	..	1	1
4.....	..	..	..
3.....	1	..	1
2.....	..	..	..
1.....	..	..	..
No data.....	..	14	14
N.....	101	290	391
Mean.....	12.9	13.1	13.0
SD.....	3.47	2.77	3.17

for 35 cents a pound. If the average weight of the sheep was 60 pounds, what was the amount of money received?"

Table 2 shows the distribution of total scores on Test 1. The maximum possible score is 22 points, made up of 10 each for relevant and irrelevant facts and 2 points for missing facts. The data for high school and for college subjects are shown separately. As indicated in the table, the mean scores of 12.9 and 13.1 for the high school and college subjects, respectively, show no significant difference between the two groups. For the total group, the range in scores was from 3 to 21, none in the

TABLE 3
NUMBER OF SUBJECTS IDENTIFYING FACTS CORRECTLY

Number identified correctly	High school			College			Total		
	Relevant	Irrelevant	Needed	Relevant	Irrelevant	Needed	Relevant	Irrelevant	Needed
10.....	2	4	..	..	10	..	2	14	..
9.....	5	9	..	14	19	..	19	28	..
8.....	17	6	..	44	13	..	61	19	..
7.....	32	2	..	83	9	..	115	11	..
6.....	19	13	..	76	41	..	95	54	..
5.....	18	23	..	43	101	..	61	124	..
4.....	6	19	..	13	51	..	19	70	..
3.....	1	19	..	..	27	..	1	46	..
2.....	..	6	33	2	5	90	2	11	123
1.....	1	..	55	1	..	162	2	..	217
0.....	..	..	13	..	..	24	..	..	37
No data.....	..	..	..	14	14	14	14	14	14
Total.....	101	101	101	290	290	290	391	391	391
Mean.....	6.5	5.2	1.2	6.5	5.4	1.2	6.5	5.3	1.2
SD.....	1.50	2.15	0.65	1.31	1.82	0.42	1.37	1.88	0.61

entire group reaching the maximum score. The wide range in scores is indicative of the confusion in thinking experienced by subjects whose reading of the problem does not lead to a clear separation of relevant facts from irrelevant.

Table 3 provides a further breakdown of the data, separating the three factors to be identified. The close resemblance between the scores of the high school and the college groups raises the problem why four additional years of education for the college group does not result in higher scores on this test, particularly in view of the often-expressed claim of colleges that they train students to think. From the table it appears that it is more difficult to recognize irrelevant facts than to recognize relevant ones. It is significant that a relatively large number of both high school and college subjects failed to recognize that needed facts were not supplied. The consequences of this failure in respect to the solving of a problem are obvious.

A still further analysis of the data on Test 1 is provided in table 4. Here, for each problem, is shown a distribution of the correct responses, expressed in percentage of the total group identifying each fact correctly. Table 4 should be read as follows: 99 per cent of the high school group (100 out of 101) identified correctly "8 men" as a relevant fact; 97 per cent (268 out of 276) of the college students identified this fact as relevant, and so on. Table 4 indicates that there is a large difference in the percentage of subjects who can identify as relevant, irrelevant, or needed, the different parts of the problems. Less than half as many students were able to recognize "number in the original crew" as the missing fact in Problem 3 as were able to recognize "number of sheep" as the missing fact in Problem 2. Likewise, in their solution of Problem 1, only 5 per cent of the high school group recognized " $\frac{1}{2}$ day" as a relevant fact, although 99 per cent of them recognized "8 men" as relevant. The matter of learning to read problems with discrimination in respect to the three kinds of facts here considered is worth serious attention. But the pertinence of the results of Test 1 for this study is that they emphasize the difficulty experienced by a subject in trying to solve a problem without first being able to distinguish relevance from irrelevance when he reads the problem.

The data for the high school group in tables 2 to 4 were drawn from the 101 cases in the Alameda group. The high school pupils from Oakland were given individually one of the problems in Test 1, but they were supplied with the missing fact in the problem and then were asked to solve it. Furthermore, they were asked to write all their computa-

TABLE 4
DISTRIBUTION OF CORRECT RESPONSES EXPRESSED IN PERCENTAGE
OF TOTAL NUMBER TAKING TEST 1

Problem and item	Relevant			Irrelevant			Facts needed		
	HS	Coll	Total	HS	Coll	Total	HS	Coll	Total
Problem 1									
1. 8 men.....	99	97	98						
2. 2 days.....	93	97	96						
3. $\frac{1}{2}$ day.....	5	7	6						
4. 160 ft.....	..	..	..	34	30	31			
5. \$1.50.....	..	..	..	88	96	94			
Problem 2									
1. 35 cents.....	97	98	98						
2. 60 lbs.....	88	87	87						
3. Jan. 14.....	..	..	..	94	95	95			
4. 17 yrs.....	..	..	..	90	97	95			
5. No. of sheep...	..	..	..	..	..	..	84	87	86
Problem 3									
1. 6 mos.....	33	36	35						
2. 800 men.....	32	32	32						
3. 1,200 gals.....	..	..	..	23	21	22			
4. 24 sacks.....	..	..	..	23	21	22			
5. 400 lbs.....	..	..	..	23	21	21			
6. 5 days.....	..	..	..	15	11	12			
7. No. in crew....	..	..	..	..	..	..	37	37	37
Problem 4									
1. 150 gals.....	94	88	90						
2. $\frac{1}{2}$ full.....	52	48	49						
3. $7\frac{1}{2}$ gals.....	56	58	58						
4. 2 acres.....	..	..	..	64	76	73			
5. $1\frac{1}{2}$ acres.....	..	..	..	65	70	69			
Number of students taking test.....	101	276	377	101	276	377	101	276	377

tions arranged in sequence on a special test blank ruled for that purpose. In addition, the experimenter kept a record of their comments as they worked. Consequently, the data from this high school group permit one further stage of analysis.

Table 5 shows the number of persons giving each type of answer to the problem, which was stated as follows: "A ship's crew uses 1,200

TABLE 5
ANALYSIS OF ANSWERS TO THE PROBLEM CONTAINING
EXTRA DATA FROM THE HIGH SCHOOL GROUP
(Oakland)

Type of answer	Number of subjects	Percentage
Correct.....	46	48
Incorrect, in months	16	17
24.....(5)*		
4.....(2)		
3.....(1)		
2.4.....(1)		
2.....(5)		
1.....(1)		
$\frac{1}{4}$(1)		
Incorrect, in days	10	10
325.....(1)		
135.....(1)		
72.....(1)		
30.....(1)		
24.....(1)		
23.....(1)		
18.....(2)		
14.....(1)		
1.2.....(1)		
Incorrect, in weeks: 6 weeks.....	1	1
Irrelevant.....	5	5
$\frac{3}{4}$ as long.....(1)		
Amount of provisions.....(4)		
No answer.....	18	19
Total.....	96	100

* The number in parentheses indicates the number of persons giving the answer.

gallons of water, 24 sacks of potatoes, and 400 pounds of meat every 5 days. The ship has provisions on hand to last her crew of 200 men for 6 months. How long would these provisions last a crew of 800 men?" As indicated in table 5, 48 per cent of the 96 pupils who took the test gave the correct answer, and 19 per cent failed to get any answer. Of the others, the various answers given are shown in table 5. From the

detailed written work of the pupils and the notes taken by the experimenter, the sequence of their thinking may be seen. Since the problem contained both relevant and irrelevant facts, the examples given below throw considerable light on the thinking processes of these pupils. The quoted examples help explain how some of the answers shown in table 5 were reached. The first three subjects all obtained the correct answer.

Subject 14: This student rejected the irrelevant data immediately. He said, "200 is to 800 as 1 is to 4... $\frac{1}{4}$ the time... $1\frac{1}{2}$ months."

Subject 3: "If the ship has provisions for 200 men for 6 months, 800 men would consume these provisions in $\frac{1}{4}$ of 6 months. Convert 6 months to years. Twenty-five hundredths of a month would be reduced into days. The answer is unreasonable. One-fourth times $6/1$ equals $3/2$ equals $1\frac{1}{2}$."

Subject 38: This student began by determining how many five-day periods there are in 1 month and then in 6 months. Next he multiplied 1,200 gal. of water, 24 sacks of potatoes, and 400 pounds of meat by 36. "That's how much food they eat in 6 months—200 men... Then you have to find out how much 800 men are going to eat in 6 months. That's 600 more... No, it says how long would these provisions last. Divide each of those by 200. That will be how much 1 man will eat. And if I multiply that by 800, that's how long... No, that isn't right... I'm trying to find out how long it will last them... It takes 6 months for 200 men. How long will it last 800 men? Just 4 times as much. Divide 4 into 6. Equals $1\frac{1}{2}$ months."

Each of these subjects arrived at the correct answer but by increasingly confused methods of thinking. Many subjects not only were confused but arrived at an incorrect answer, as illustrated below.

Subject 8: This student first added the amounts of provisions and then multiplied by 30 (the number of days per month). Then he multiplied the total provisions by 5 (days). This product he then multiplied by 6 (months). He then said, "This is not right," and guessed at the answer, giving 3 months.

Subject 31: This student first divided the 1,200 gallons of water, 24 sacks of potatoes, and 400 pounds of meat by 5 (the number of days), and then added the 3 quotients, getting 306. She then subtracted 200 (the number of men) from 800 (men). Then she subtracted the provisions needed for 200 men from the amount needed by 800 men and divided by 180, the number of days in 6 months. With numerous computational errors en route, she finally arrived at the erroneous answer "325 days."

The Thurstone test of Primary Mental Abilities was taken by 96 of the 108 subjects in this group. A comparison of the total scores on this test with the reactions of the pupils to the extra facts in the problem throws some light on the results. Forty-six per cent of the 96 subjects rejected the extra data and proceeded to solve the problem using only the pertinent facts. The mean total unweighted score of the group on the Primary Mental Abilities Test was 135 points. Eighteen

per cent of the subjects first attempted to use the extra facts, then saw their error and solved the problem without them. The mean mental score of this group was 138. Thirty-six per cent of the subjects attempted to use the extra, irrelevant data throughout their work. The mean mental-ability score of this group was 114 points. For the subjects in this study, lower mental ability could be one of the factors related to lack of discrimination between relevant and irrelevant facts in a problem.

The purpose of Test 1 was to find how the subjects approached problems which contained extra facts or failed to contain needed facts. As was shown in table 2, the mean subject in the total group succeeded in identifying 13 of the 22 such facts in the problems, and approximately two-thirds of the group varied from the mean by not more than 3 facts. As was indicated in table 3, it was slightly easier to recognize relevant facts than irrelevant ones, and needed facts were more difficult to identify, although it should be noted that the sample of needed facts was small. Table 4 showed great irregularity in response, some facts being much more difficult for the pupils to identify than others. When more detailed data were secured from the high school group reported in table 5, it was evident that failure to recognize relevance and irrelevance produced great variation in ways of solving the problem and in the answers obtained. The over-all picture from Test 1 indicates that wide variations in problem solving arise from inadequate reading of the problems and from failure to identify correctly at the start just what are the relevant, irrelevant, or needed facts.

TEST 2, ESTIMATING ANSWERS

One of the commonly recommended procedures preliminary to solving a problem is to estimate the answer. Such an estimate if reasonably accurate, should add to the general understanding of the problem and serve to prevent the type of gross error in the answer which sometimes results from misplacing a decimal point or from incorrect computation. Test 2 contained four problems such as the following: "A machine that sold regularly at \$985 was priced at a 10 per cent discount. What was the discounted price?" The subject was asked to write the approximate answer in the space at the right and then to explain how he made the estimation. The expected procedure for the example just cited was as follows: \$985 is about \$1,000. 10 per cent of \$1,000 is \$100. \$1,000 minus \$100 = \$900."

It should be noted that the "expected procedure" is not the only one

TABLE 6
DEVIATION FROM EXPECTED APPROXIMATE ANSWERS: TEST 2

Problem	Zero deviation			± 1-10 per cent			± 11-25 per cent			± 26 per cent or more			Total cases		
	HS	College	Total	HS	College	Total	HS	College	Total	HS	College	Total	HS	College	Total
1.....	3	23	26	62	180	242	3	9	12	31	59	90	99	271	370
2.....	59	152	211	26	77	103	10	13	23	6	30	36	101	272	373
3.....	49	101	150	34	47	81	7	14	21	11	24	35	101	186	287
4.....	42	77	119	27	47	74	2	13	15	29	48	77	100	185	285
Total.....	153	353	506	149	351	500	22	49	71	77	161	238	401	914	1315
Percentage.	38	39	39	37	38	38	6	5	5	19	18	18	100	100	100

that would produce a useful approximation. It is only what seems to the writer to be the easiest and quickest approximation. For example, an equally valid method for Problem 1 would be as follows: "\$985 is about \$990. 10 per cent of \$990 is \$99 (or less than \$100). $\$990 - \$99$ (or \$100) = \$890." By referring ahead to table 8, Problem 1, one can see that methods 12 and 13 produced \$890 as the approximate answer, and that 10 high school and 5 college subjects gave one or the other of these two answers. The data in table 6 for Problem 1 are based on the "expected approximation" of \$890. In Problem 1, particularly, several other methods of approximating would produce answers only slightly different and equally useful. As stated earlier, this study is not concerned with arithmetical procedures *per se* but rather with variations in methods of thinking. The data in table 8 show clearly how great this variation is.

Table 6 shows the results of Test 2. The high school group shown here was made up of the 101 pupils from Alameda. The Oakland group was given a modified form of the test individually and the results will be reported in table 7. Some of the college subjects were given only the first two problems of the test. As shown in table 6, 39 per cent of these subjects gave the expected approximation (zero deviation), and an additional 38 per cent gave an answer that varied from the expected one by not more than 10 per cent, a deviation due mainly to the rounding of the numbers before the estimate was made. It will be noted from the table that 18 per cent of all subjects varied from the expected answer by 26 per cent or more. Approximately one-fifth of the entire group made estimates that were too far from the expected answer to be of any great help.

Table 7 gives an analysis of the methods used in estimating. This table was derived from data obtained from a group of Oakland High School pupils who were individually given one problem from Test 2. As will be noted, 51 per cent used exact numbers and exact methods of computing. Although they did this mentally rather than with pencil and paper, this method requires more time than a quick estimate and is of little, if any, more value. Only 16 per cent of the subjects used wrong processes or guessed. However, there were many who failed to take advantage of rounding numbers or to use effective methods for rapid approximation.

The variety of procedures employed in estimating is shown in detail in table 8 for Problems 1 and 2.⁸ The categories here were derived from

⁸ Problem 2 is as follows: "At an auction sale, 98 books sold for a total of \$497.50. What was the average amount received per book?"

TABLE 7
METHOD USED FOR APPROXIMATING ANSWERS BY SUBJECTS

Method of approximating	Fre- quency	Per- centage
Used exact method and exact numbers, with mental calculations	43	51
Used exact method but approximate (rounded) numbers.....	11	13
Used simplified method but exact numbers in all but one case..	16	19
Used wrong or incomplete process with exact numbers.....	12	14
Guessed at answer.....	2	2
Total.....	84	99

the written explanations of 101 high school and 108 college subjects. A study of this table shows that although there is some agreement on methods of approximating, there is far more variation than agreement. Many of the subjects do not avail themselves of the advantage afforded by a quick estimate of the answer in solving problems.

TEST 3, PROBLEM-SOLVING SEQUENCE

When the process of solving a problem involves a number of operations, the sequence of performing these operations may have some effect on the solution. Test 3 was designed to provide data to show the relationship between the sequence or pattern of operations and the accuracy of the solution.

The problems in Test 3 are alike in general structure. Each contains four facts needed for solving the problem, and its solution involves three consecutive multiplications. The problems differ only in the language and numbers used and in the order of stating the facts. This part of the study is significant only as it develops a procedure for study of the more difficult problem treated in chapters 4 and 5. The structure of the problems is illustrated by the first in the series: "If 15 hens lay 5 eggs per week for 4 weeks, what is the total value of the eggs at 6 cents each?" Space was left below each problem, and the subjects were instructed to "write what you do first in the first space, what you do next in the second space, and so on." From these consecutive blocks it was possible to determine the sequence of operations.

Each problem contained four elements, which are represented by the letters *a*, *b*, *c*, and *d*. The elements of the problems were as follows:

	<i>Element a</i>	<i>Element b</i>	<i>Element c</i>	<i>Element d</i>
Problem 1.....	15 hens	5 eggs	4 weeks	6 cents
Problem 2.....	12 boys	8 hours	5 days	60 cents
Problem 3.....	20 rows	24 seats	3 shows	50 cents

TABLE 8

ANALYSIS OF METHODS OF APPROXIMATION, HIGH SCHOOL AND COLLEGE GROUPS: TEST 2, PROBLEMS 1 AND 2

Method of approximation	Number of subjects	
	High school (Alameda)	College (Fall 1952)
PROBLEM 1		
I. <i>Expected approximation (\$900.00):</i>	(3)	(11)
1. \$985 is about \$1,000. Ten per cent of \$1,000 is \$100. \$1,000 - \$100 = \$900.....	1	4
2. Ten per cent of \$985 = \$98.50. \$985 - \$98.50 equals about \$900.....	1	3
3. "I think it would be \$900 because 10 per cent discount of \$100 would be \$90".....	1	..
4. Ten per cent off \$1,000 is \$900.....	..	2
5. \$985 - \$85 = \$900.....	..	1
6. Ten per cent of \$985 is about \$90. \$985 - \$90 = about \$900.....	..	1
II. <i>Answer $\pm$ 10 per cent from expected approximation:</i>	(62)	(63)
7. (a) Ten per cent of \$985 is \$9.00. \$985 - \$9.00 equals \$976. (b) Ten per cent of \$985 is \$9.85. \$985 - \$9.85 = \$975.15. (c) Ten per cent of \$985 is \$10.00. \$985 - \$10.00 = \$975.....	13	12
8. Same as no. 2 except that the exact difference of \$886.50 was given.....	14	4
9. Same as no. 8 except that the figure given as the "exact" difference was incorrect.....	9	5
10. Ten per cent of \$985 = \$98.00. \$985 - \$98 = \$887....	9	10
11. Ten per cent of \$985 is about \$100. \$985 - \$100 = \$885.	4	15
12. Same as no. 11, with answer rounded to \$890.....	5	3
13. Same as no. 8, with answer rounded to \$890.....	5	2
14. Same as no. 11, plus "a little more" = \$898.....	1	..
15. Same as no. 11, with the discount "slightly less than \$100" = \$880.....	..	2
16. "I imagined the \$985 as 100 per cent, and subtracted accordingly." Answer: \$895.....	1	..
17. No explanation. Answer: \$925.....	1	..
18. Same as no. 1, with answer "slightly less than \$900," e.g. \$892 or \$895.....	..	2
19. Ten per cent of \$985 is about \$100. \$985 - \$100 = \$885, plus 2 = \$887.....	..	3
20. Ten per cent of \$985 = about \$99. \$985 - \$99 = \$886.	..	2
21. Ten per cent of \$985 = \$90; therefore, 10 per cent of \$985 = about \$850.....	..	1

TABLE 8—Continued

Method of approximation	Number of subjects	
	High school (Alameda)	College (Fall 1952)
22. Ten per cent of \$1.00 = \$.10. Then 10 per cent of \$985 = \$98.50.....	..	1
23. \$985 ÷ 10 per cent = \$980.....	..	1
III. Answer ± 25 per cent from expected approximation:	(3)	(4)
24. "I took the approximet 10% added and instead subtracted it." (Spelling as in original.) Answer: \$725.	1	..
25. "By taking 10 per cent off of each dollar." Answer: \$800.....	1	..
26. Same as no. 2, giving the answer as about \$715.....	1	..
27. Same as no. 11, "rounding" to \$800.....	..	1
28. Same as no. 2, giving the answer as about \$800.....	..	3
IV. Answer ± 26 per cent (or more) from expected approximation:	(31)	(22)
29. Same as no. 7. Answers: \$187.50, \$88.50, \$89.00.....	3	..
30. "1/10 of approx. 99 dollars is \$9 subtracted from 99 and you get the answer plus small amounts according to the extra five dollars." Answer: \$89.....	1	..
31. Ten per cent of \$985, or move decimal 1 place to the left. Answer: \$98.50.....	18	12
32. Ten per cent of \$985, or move decimal 2 places to the left. Answer: \$9.85.....	6	6
33. \$985 is about \$1,000. Ten per cent of \$1,000 equals \$100.	2	..
34. Same as no. 33, with answer approximated at \$8.00...	1	..
35. Same as no. 31. Answer: \$8.50.....	..	1
36. "Crossed off 1 zero." Answer: \$9,850.....	..	1
37. "Multiply the figure by ten, or simply add zero." \$985,000.....	..	1
38. No explanation. Answer: \$90.85.....	..	1
No data.....	2	8
Total.....	101	108
PROBLEM 2		
I. Expected approximation (\$5.00):	(59)	(61)
1. Ninety-eight books are about 100; \$497.50 is about \$500. \$500 ÷ \$100 = \$5.00.....	22	34
2. \$497.50 ÷ \$100 = \$5.00 in rounded numbers.....	3	8
3. \$497.50 ÷ \$98 = \$5.00 in rounded numbers.....	33	14
4. \$490 ÷ \$90 = \$5.00.....	..	2

TABLE 8—Continued

Method of approximation	Number of subjects	
	High school (Alameda)	College (Fall 1952)
5. Select a number which appears to be a reasonable solution and multiply it by 98.....	..	3
6. "Half of 98 is approximately 490, leaving approximately \$5 for each.".....	1	..
II. Answer ± 10 per cent from expected approximation:	(26)	(24)
7. Same as no. 3, giving an exact answer (including arithmetically incorrect answers).....	12	5
8. Same as no. 3, giving estimated solutions extrapolated from their beginning calculations, e.g., \$4.50, \$5.50, \$5.25.....	4	2
9. Same as no. 2, rounded to \$4.99.....	..	1
4.98.....	4	3
4.97.....	3	1
4.96.....	..	1
4.90.....	..	2
4.50.....	..	3
10. Same as no. 1, plus or minus "a little"		
Answer: \$5.08.....	1	..
4.98.....	..	1
4.75.....	2	1
4.50.....	..	1
11. Same as no. 4, giving answer as \$5.50.....	..	1
12. "Divided mentally" or process unexplained		
Answer: \$4.50.....	..	1
4.90.....	..	1
III. Answer $\pm 11-25$ per cent from expected approximation:	(10)	(7)
13. Same as no. 2, rounded to \$4.00.....	..	2
14. Same as no. 3. Answer: \$4.00.....	..	1
15. Same as no. 4. Answer: \$6.00.....	..	1
16. Same as no. 7, with arithmetic errors.		
Answer: \$5.80.....	2	1
5.75.....	..	1
5.70.....	3	1
5.60.....	1	..
17. Same as no. 8. Answer: \$4.00, \$4.12, \$4.25, etc.....	4	..
IV. Answers ± 26 per cent (or more) from expected approximation:	(6)	(9)
18. Same as no. 1. Incorrect answer: \$.50.....	..	1
50.00.....	..	2
19. Same as no. 2. Incorrect answer: \$49.00.....	..	1

TABLE 8—*Concluded*

Method of approximation	Number of subjects	
	High school (Alameda)	College (Fall 1952)
20. Same as no. 4. Incorrect answers: \$50.00.....	..	1
21. Same as no. 7. Incorrect answers: \$56.40, \$57.00, \$44.80.....	5	2
22. Same as no. 8. Answer: \$0.65.....	..	1
23. Ninety-eight is about 100 books. Multiply 497.50 by 100 by moving decimal one place to the right: \$4,975.00.....	..	1
24. Same as no. 9. Answer: \$497.50.....	1	..
No data.....	..	7
Total.....	101	108

The elements were expressed in different sequence in each problem, as follows:

	<i>Sequence of Elements</i>			
	<i>1st</i>	<i>2d</i>	<i>3d</i>	<i>4th</i>
Problem 1.....	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>
Problem 2.....	<i>d</i>	<i>a</i>	<i>b</i>	<i>c</i>
Problem 3.....	<i>a</i>	<i>b</i>	<i>d</i>	<i>c</i>

The purpose of the analysis which follows is to find whether there are common patterns of solving the problems and, if so, whether they are related to the order in which the elements are expressed or to the logic of the solution. The data for 101 high school subjects in solving Problem 1 are shown in figure 1. The reader should keep in mind that the letter symbols refer to the facts expressed above and that the chart should be read from left to right. The number of subjects is shown above the rectangles and operations are shown within.

Of the 101 high school subjects shown in figure 1, 81 began by multiplying *a* (the number of hens) by *b* (the number of eggs). Seventy-nine of this group of 81 then multiplied the product obtained by *c* (the number of weeks); and the other 2 introduced an extra element (*x* = number of days per week), multiplying *c* by *x*. Of the 79 students who multiplied *c* (*ab*), 2 then divided (*abc*) by *d* (number of cents), and the remaining 77 multiplied (*abc*) by *d*. By following the lines across the chart, one may see the sequences of operations employed in solving this problem.

Figure 2 shows in similar form the operations employed by this same group of high school pupils in solving Problem 2. Here the variety of

Read left to right.

Elements:

- a. 15 hens
- b. 5 eggs
- c. 4 weeks
- d. 6 cents
- (x. days per week)

Problem: If 15 hens each lay 5 eggs per week for 4 weeks, what is the total value of the eggs at 6 cents each.

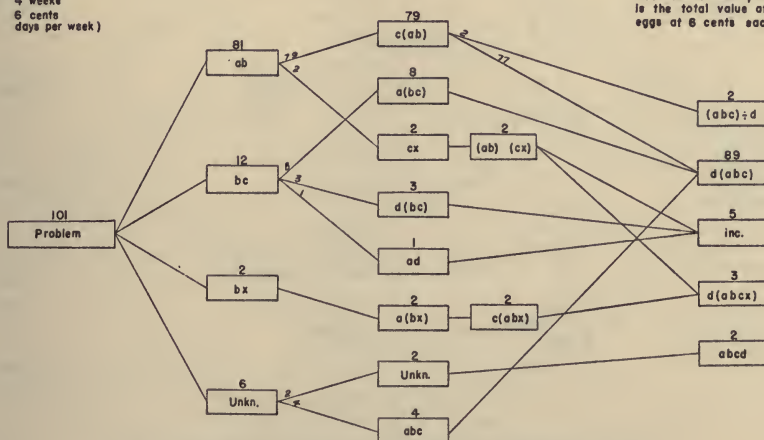


Fig. 1. Patterns of problem-solving thinking of 101 high school students on Problem 1, Test 3.
Number of subjects above rectangles. Operations within rectangles.

Read left to right.

Elements:

- a. 12 boys
- b. 8 hours
- c. 5 days
- d. 60 cents

Problem: At 60 cents per hour find the total earnings of 12 boys if they each worked 8 hours per day for 5 days.

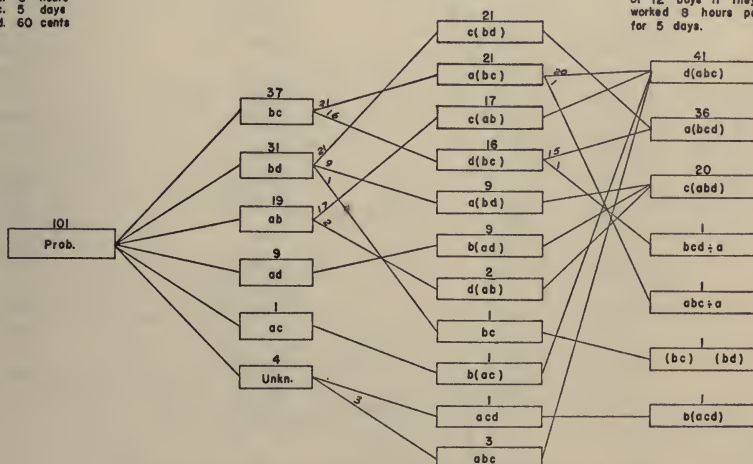


Fig. 2. Patterns of problem-solving thinking of 101 high school students on Problem 2.
Number of subjects above rectangles. Operations within rectangles.

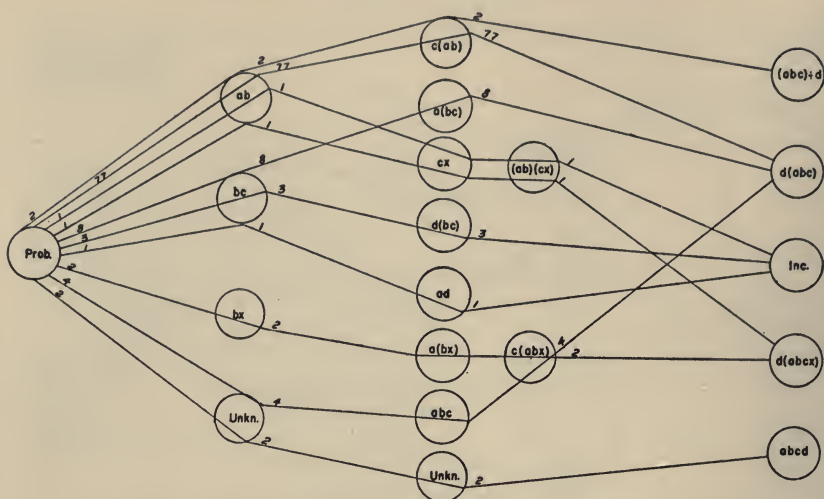


Fig. 3. Complete sequences of problem solving for 101 high school students on Problem 1.
Numbers on lines indicate number of subjects following each pattern.

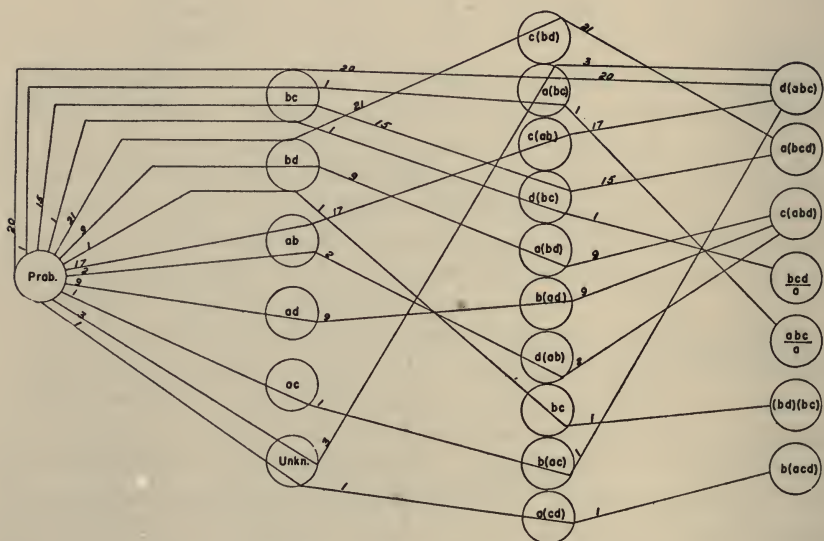


Fig. 4. Complete sequence of problem solving for 101 high school students on Problem 2.
Numbers on lines indicate number of subjects following each pattern.

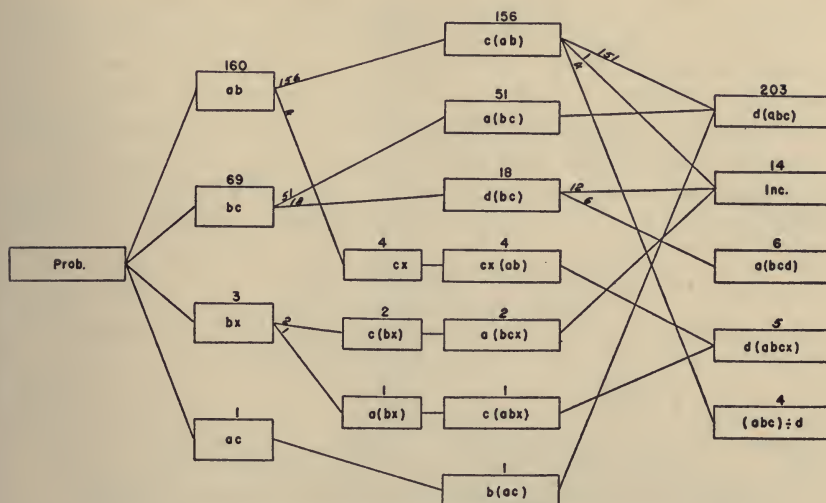


Fig. 5. Patterns of problem-solving thinking of 233 college students on Problem 1, Test 3. Number of subjects above rectangles. Operations within rectangles.

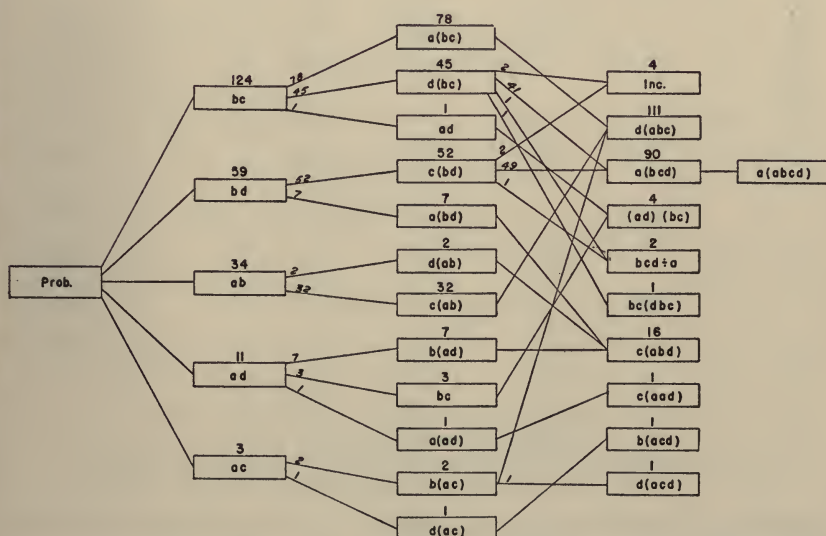


Fig. 6. Patterns of problem-solving thinking of 231 college students on Problem 2, Test 3. Number of subjects above rectangles. Operations within rectangles.

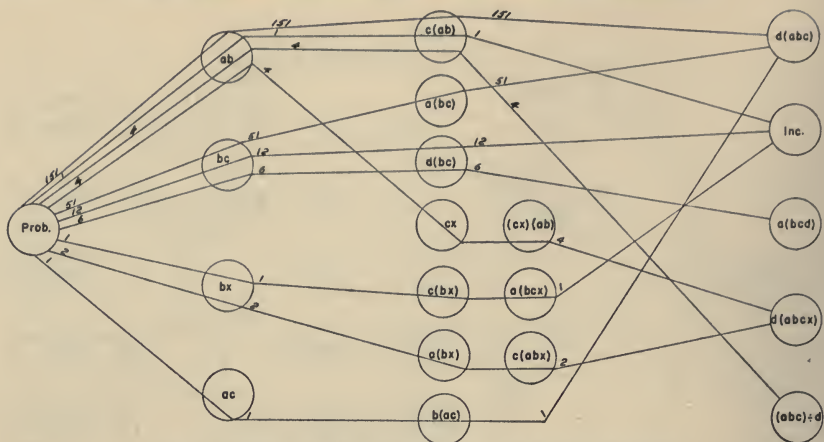


Fig. 7. Complete sequences of problem solving for 233 college students on Problem 1, Test 3. Numbers on lines indicate number of subjects following each pattern.

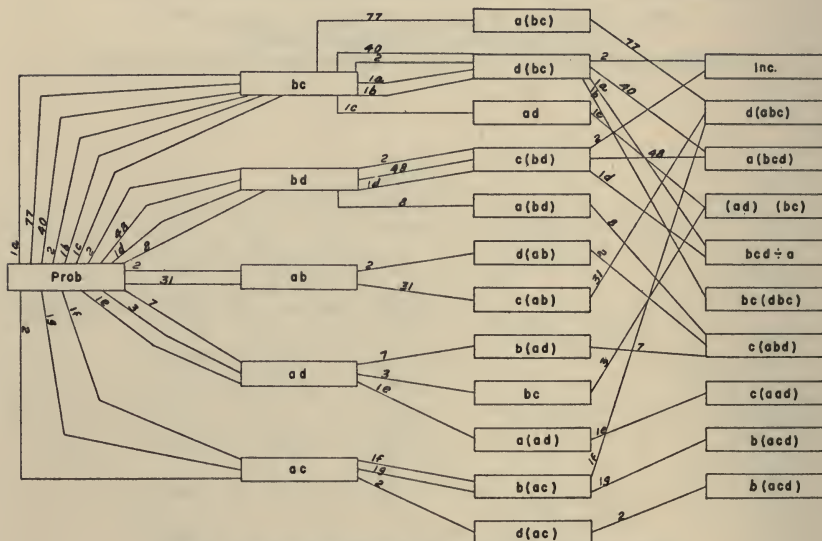


Fig. 8. Complete sequence of problem solving for 229 college students on Problem 2. Numbers on lines indicate number of subjects following each pattern. (Two cases with confused sequence not included.)

operations is greater than for Problem 1, and no single sequence of operations stands out as most preferred.

The sequences employed in solving the problems appear more clearly if lines are drawn to show single groups of subjects that follow an identical pattern across the chart. Figures 3 and 4 show the same data

as were presented in figures 1 and 2, but in figures 3 and 4 each line represents a single group of subjects. For example, in figure 3, 77 subjects followed the sequence: $ab-c(ab)-d(abc)$. The remaining 24 subjects followed 9 different sequences, the greatest number in any one sequence being 8. Figure 4 shows the same kind of facts for Problem 2.

Data from 233 college students who took Test 3 were available. Figures 5 to 8 show the same kind of facts about the college group as figures 1 to 4 did about the high school group. The patterns of solution

TABLE 9
SEQUENCE OF OPERATIONS: TEST 3

Section a								
Group	Problem no.	No. of cases	Operation			Answer		
			1	2	3	Wrong	Right	Percentage right
High school (101 cases)	1	77	ab	c(ab)	d(cab)	11	66	86
		24	(in 9 different sequences)			11	13	54
	2	21	bd	c(bd)	a(cbd)	4	17	81
		20	bc	a(bc)	d(abc)	1	19	95
		17	ab	c(ab)	d(cab)	0	17	100
		15	bc	d(bc)	a dbc)	2	13	87
		28	(in 9 different sequences)			8	20	71
	3	84	ab	d(ab)	c(dab)	15	69	82
		10	ab	c(ab)	d(cab)	0	10	100
		6	(in 5 different sequences)			0	6	100
		1	(omitted problem)					
College (233 cases)	1	151	ab	c(ab)	d(cab)	10	141	93
		51	bc	a(bc)	d(abc)	2	49	96
		31	(in 8 different sequences)			26	5	16
	2	77	bc	a(bc)	d(abc)	3	74	96
		48	bd	c(bd)	a(cbd)	5	43	90
		40	bc	d(bc)	a dbc)	1	39	98
		31	ab	c(ab)	d(cab)	4	27	87
		35	(in 14 different sequences)			19	16	46
		2	(omitted problem)					
	3	161	ab	d(ab)	c(dab)	20	141	88
		49	ab	c(ab)	d(cab)	3	46	94
		22	(in 11 different sequences)			7	15	68
		1	(omitted problem)					

TABLE 9—*Continued*

Section <i>b</i>					
Group	Problem	Number of cases	Sequence of operation		
			1	2	3
High school	1	77	ab	c	d
	2	21	bd	c	a
		20	bc	a	d
		17	ab	c	d
		15	bc	d	a
	3	84	ab	d	c
		10	ab	c	d
College	1	151	ab	c	d
		51	bc	a	d
	2	77	bc	a	d
		48	bd	c	a
		40	bc	d	a
		31	ab	c	d
	3	161	ab	d	c
		49	ab	c	d

exhibited by the high school and by the college groups show no marked differences.

Table 9 shows the patterns of solution used by both groups of subjects and for all three problems, except the patterns representing fewer than ten cases, which are not shown. For example, there were 24 high school subjects who displayed 9 different patterns of thinking in trying to solve Problem 1, but no one pattern was used by more than 8 students. Among the high school subjects there were 9 such small-group patterns of thinking on Problem 2, and 5 different patterns on Problem 3. Among the college subjects there were 8 different small-group sequences of thinking on Problem 1, and 14 and 11, respectively, on Problems 2 and 3. The point of interest here is not so much that there is variety in the patterns as it is that the same sequence of operations was followed by so many subjects. The order of appearance of the four elements of each problem is recapitulated in table 9, section *b*, and should be compared with the sequence in the verbal statement of the problems shown on page 144.

In Problem 1, element *d* occurs last in the statement of the problem and most frequently last in the sequence of operations. In Problem 2, element *d* occurs first in the statement of the problem but occurs more often in the last than in the first operation. In Problem 3, element *d* appears third in the problem but occurs more often in the second than in the third operation. On the whole, the logic of the operations seems to exert more weight than the order of statement in the problem.

Since it is clear that some patterns of solution are more common than others, the question arises whether some patterns bring more right answers than others do. Evidence bearing on the answer to this question is given in the last three columns of table 9, section *a*. The logic of the problem and the computations involved are so simple that few errors are made by high school and college subjects. Nevertheless, it is worth noting that, with the exception of the patterns followed by the high school subjects who worked on Problem 3, the methods employed by the greater number of subjects produced a higher percentage of correct answers than the less frequently used sequences which are grouped together. This question of the relative accuracy attained with the different sequences used in solving problems will be a matter of major interest when applied to the difficult problems analyzed in chapter vi. The principal fact disclosed by Test 3 is that although there appears to be a considerable degree of uniformity in the sequence employed by a group of subjects in solving a simple problem, there is also a great deal of individual variation in patterns of solution, as shown in table 9.

TEST 4, USING NUMBERS AND LETTER SYMBOLS

Test 4 was prepared for the purpose of studying the difference in the methods used in solving problems when the facts are expressed in numbers or in letter symbols. In Test 4 there are three problems (nos. 2, 3, and 5) which are similar to Problems 4, 6, and 7 except that numbers are used in the first three, and letter symbols in the last three. The over-all results from the test are shown in table 10.

As shown in table 10, the high school subjects do somewhat better when data are expressed in numbers rather than in letter symbols, the percentage getting all three problems correct being 34 per cent when numbers are used, and 28 per cent when letters are used. The college subjects show no significant difference, the percentages being 24 and 23. Taking the group as a whole, 27 per cent solved all three number problems correctly and 25 per cent solved all the letter-symbol problems correctly. However, some subjects omitted one or more of the

TABLE 10

COMPARISON OF RESULTS ON PROBLEMS EXPRESSED IN NUMBERS AND IN LETTER SYMBOLS

Number of problems correct	Number						Letter symbols					
	HS			College			HS			College		
	Frequency		Per-centage	Frequency		Per-centage	Frequency		Per-centage	Frequency		Per-centage
	Frequency	Per-centage	Per-centage	Frequency	Per-centage	Per-centage	Frequency	Per-centage	Per-centage	Frequency	Per-centage	Per-centage
3.....	32	34	24	41	24	27	27	28	23	40	67	25
2.....	27	28	40	70	40	36	38	40	34	59	97	36
1.....	31	33	25	44	25	28	17	18	28	49	66	25
0.....	5	5	11	19	11	9	13	14	15	26	39	14
Total.....	95	100	100	174	100	100	95	100	100	174	269	100

TABLE 11

ANALYSIS OF PAIRS OF NUMBER AND LETTER-SYMBOL PROBLEMS IN WHICH ONE OR BOTH PROBLEMS WERE OMITTED

	Numbers (Problems 2, 3, 5)						Letter symbols (Problems 4, 6, 7)					
	High school			College			High school			College		
	Frequency		Per-centage	Frequency		Per-centage	Frequency		Per-centage	Frequency		Per-centage
	Frequency	Per-centage	Per-centage	Frequency	Per-centage	Per-centage	Frequency	Per-centage	Per-centage	Frequency	Per-centage	Per-centage
Right answers.....	3	13	29	16	29	24	..	..	..	..	..	..
Wrong answers.....	11	46	33	18	33	37	3	13	4	5	6	6
Omitted.....	10	42	38	21	38	39	21	88	96	74	94	94
Total no. of pairs...	24	101	100	55	100	100	24	101	100	79	100	100

problems and hence were not recorded in table 10, which includes only those who attempted all three problems of both kinds.

The problems in the number- and letter-symbol series were paired; for example, Problem 2 (numbers) was like Problem 4 (letter symbols). Table 11 presents an analysis of the subjects who omitted either one or both problems in a pair. There were 24 such incompleting pairs among the high school students tested and 55 among the college students—79 such pairs in all.

As shown in table 11, 3 high school students and 16 college students gave right answers to the number problems but either gave a wrong answer or omitted the corresponding letter-symbol problems of the pairs. It is clear that there was much more difficulty with the letter-symbol problems than with those expressed in numbers. In 94 per cent of the 79 pairs, the letter-symbol problem was not answered, and in 39 per cent, the number problem. In no case was a letter-symbol problem in a pair solved correctly. In only 6 per cent of the pairs was an incorrectly answered letter-symbol problem matched by a number problem that was unanswered. For many students, it was apparent that their previous education had resulted in a lack of confidence in their ability to deal with letter symbols.

SUMMARY

Table 12 shows the distribution of total scores for all four sections of the test combined and for all persons taking the tests except the 108 high school students who were given, individually, only two problems from Tests 1 to 4. The table illustrates the usual wide spread of individual differences.

To whatever extent problem solving is affected by the recognition of relevant, irrelevant, and needed data, by the practice of estimating answers in advance, by the use of an effective sequence of operations, and by the ability to think with both numbers and letter symbols, it is apparent that some persons have great advantage over others. These are learned factors, and to a considerable degree they reflect the habits and practices that have resulted from instruction in schools. To be able to identify relevant facts, to make an estimate of a reasonable answer, and to use effectively both numbers and letter symbols undoubtedly are helpful. But the central problem of this study is to find what processes of thinking are employed in the actual solution of a problem. The following chapters will attempt to penetrate beyond the external or preliminary factors dealt with in Tests 1 to 4 and to study the nature of thinking when the purpose is to discover a generalization and when it is to solve a perplexing problem.

TABLE 12

DISTRIBUTION OF TOTAL SCORES FOR THE COLLEGE AND
HIGH SCHOOL GROUPS: TESTS 1-4

Total score	Frequency		
	High school	College	Total
35.....	..	1	1
34.....	..	1	1
33.....	..	2	2
32.....	2	0	2
31.....	4	4	8
30.....	1	1	2
29.....	2	8	10
28.....	8	8	16
27.....	1	6	7
26.....	4	14	18
25.....	6	10	16
24.....	3	9	12
23.....	14	13	27
22.....	7	24	31
21.....	3	20	23
20.....	6	16	22
19.....	7	27	34
18.....	5	22	27
17.....	4	11	15
16.....	4	13	17
15.....	4	7	11
14.....	3	3	6
13.....	2	2	4
12.....	4	0	4
11.....	1	2	3
10.....	0	0	0
9.....	0	1	1
8.....	0	..	0
7.....	1	..	1
6.....	..	..	..
5.....	..	..	..
4.....	..	..	..
3.....	..	..	..
2.....	..	..	..
1.....	..	..	..
0.....	..	..	..
No data.....	5	65	70
Total.....	101	290	391
M.....	21.5	21.4	21.4
SD.....	5.43	4.51	4.80

CHAPTER III. DISCOVERING GENERALIZATIONS

There are types of problems which for their solution require the discovery of a generalization. The ability to perceive a common factor in a number of situations is an important characteristic of problem-solving thinking. Test 5 in the series used for this study is concerned with the process of discovering a generalization.

Three aspects of generalizing are considered in this chapter. The first deals with the questions how much time and how many specific experiences are required for discovering a generalization. The second aspect is the level of generalization that is discovered, and the third has to do with the ability of a subject to transfer a generalization to another problem to which it is applicable. The subject matter of the test is a number of columns of even and odd numbers in series, and the problem is to discover a rule which would enable one to find the sum of a series of numbers more quickly than by the usual process of addition. The specific directions for the exercises in the first problem were as follows. "Can you find the sum of a series of numbers (such as 14, 15, 16, 17, 18, 19, 20) more quickly than by adding the numbers in sequence? Find the sum of exercises a to k and write the answer under each column. When you have finished exercise k , write the method that you have discovered." There then followed eleven (a to k) exercises such as the ones below:

a	b	c	d	e	f
1	4	3	2	1	22
2	5	4	3	2	23
3	6	5	4	3	24
		6	5	4	25
		7	6	5	26
			7	6	27
			8	7	28
				8	
				9	

Below the exercises, space was provided for writing the generalization that was discovered.

The exercises in Problem 2 differed from those in Problem 1 in two respects. All columns began with the number 1 and contained only odd numbers, and at the top of each column the number of odd numbers in the column was given in parentheses.

Problem 3 was designed to test the transfer of the generalization covering the sum of a series of odd numbers by applying it in finding

TABLE 13

TIME IN SECONDS FOR OBTAINING SUMS IN PROBLEM 1 FOR 18 COLLEGE STUDENTS WHO REACHED A CORRECT GENERALIZATION

Subject number	Column									
	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	<i>i</i>	<i>j</i>	<i>k</i>
8.....	13	11	17	10	33	37	27	24	64	27
10.....	2	4	8	9	15	20	21	24	32	14
16.....	2	18	11	14	34	35	19	9	20	18
28.....	2	4	5	10	16	15	16	15	24	13
30.....	2	8	8	20	19	31	42	37	58	36
39.....	3	4	5	8	12	18	20	34	151	27
40.....	3	9	12	16	17	29	40	34	60	26
41.....	3	7	7	13	21	31	37	144	201	150
42.....	4	7	14	9	13	15	23	16	28	18
43.....	3	55	46	45	84	173	63	69	101	88
49.....	4	23	13	109	37	62	445	36	36	43
51.....	3	8	22	20	29	29	34	18	53	33
52.....	4	10	12	10	31	70	24	25	84	12
56.....	6	11	16	26	37	50	47	102	79	35
59.....	6	8	8	9	11	13	8	10	10	33
62.....	2	16	15	28	18	28	25	20	19	15
63.....	2	5	7	14	19	35	33	30	30	29
66.....	6	13	14	20	25	57	44	29	52	86

Tables such as the one given above were constructed for Problem 2 and also for groups of students who failed to make a correct generalization on both Problems 1 and 2, but in no case was there a consistent pattern of time durations. A differently constructed set of columns and a different method of recording which would separate time used in finding the sums from time used in thinking about possible generalization might afford a better test of the hypothesis. The idea of a time analysis seems to be worth further research.

LEVEL OF GENERALIZATIONS DISCOVERED

As stated previously, each subject was asked to formulate a generalization in writing, at the end of each of the three problems. The results of their efforts ranged from a correctly stated rule covering the case to a complete lack of ability to formulate any generalization. A classification of the generalizations of 69 college students who were given the test individually resulted in seven different generalizations for Problem 1, two additional generalizations for Problem 2, and four more for

Problem 3. These thirteen generalizations are given in the following list, and in subsequent tables they are referred to by these classification numbers.

CLASSIFICATION OF GENERALIZATIONS FOR TEST 5

PROBLEM 1

- Type 1.* Multiply the middle number in the series by the number of terms in the series if there is an odd number of terms. If an even number of terms, use the number halfway between the two middle numbers (or the average of the two middle numbers).
- Type 2.* For a series with an odd number of terms, multiply the middle number in the series by the number of terms in the series. If there is an even number of terms, subtract one term at each end of the series and add this remainder to the total obtained by using the rule.
- Type 3.* Multiply the sum of the first and last terms by one-half the number of terms in the series.
- Type 4.* The sum of the pair of numbers at the extremes, the second pair of numbers, the third pair of numbers, and so forth, are equal. Total the sums of these corresponding pairs of numbers and then add the middle number. If the series has an even number of terms, subtract one term and add it later to the sum obtained by the rule.
- Type 5.* If there is an even number of terms proceed as in Type 4 for an odd number of terms, but there will be no middle number to add.
- Type 6.* Multiply the first number in the series by the number of terms in the series and add the number in excess of the first number for each term in the series.
- Type 7.* No valid generalization. The subject found the sums by adding.

PROBLEM 2

Each of the above-listed types except number 6, plus the following:

- Type 8.* Square the number of terms in the series.
- Type 9.* The middle number equals the number of terms in the series; therefore, either square the middle number of the series or square the number of terms.

PROBLEM 3

- Type 10.* Consider the pyramid as a series of odd numbers and use generalization number 8 or 9.
- Type 11.* Transform the pyramid into a square with 9 x 's on each side.
- Type 12.* Divide the pyramid into two series of numbers on either side of the center column. Then use the general rule developed in Problem 1 and add to get the total.
- Type 13.* The subject counted or added, or used a generalization without being able to explain why it is applicable.

Table 14 shows the number of college student subjects giving each type of generalization for Problem 1. Thirty of the students formulated a generalization independently, as indicated in the table. The remaining students were able to reach a generalization only with the help of certain clues given by the experimenter. Nine were able to do this after one clue, 17 more after a second clue, and 13 more after a third clue. The clues supplied were as follows:

Clue number 1: Look at the structure of each series. Notice that the numbers are in sequence, with each number increasing by one.

Clue number 2: Notice that there is a point in each series at which there are just as many numbers before as there are after it. Also note that the numbers after this point *increase* in the same way that the numbers before it *decrease*. (If necessary, this clue was illustrated, using column *d*.)

Clue number 3: Notice that when you add corresponding numbers before and after the mid-point the sums are equal.

TABLE 14
TYPES OF GENERALIZATIONS FORMULATED FOR PROBLEM 1
BY 69 COLLEGE STUDENTS

Type number	Without help of clues	With first clue	With second clue	With third clue	Total
1.....	4	3	7	0	14
2.....	12	1	6	1	20
3.....	1	1	1	0	3
4.....	7	1	0	6	14
5.....	1	0	2	4	7
6.....	0	3	0	0	3
7.....	5	0	1	2	8
Total.....	30	9	17	13	69

Generalizations 1, 2, and 3 were considered satisfactory.

Generalizations numbers 1 to 7 are arranged in a decreasing level of values. Generalization number 1 is the best. Number 2, although correct, is awkwardly stated and requires special treatment for one term in the series when there is an even number of terms. Generalization number 3 is correct but not quite so easy to use as number 1. Numbers 4 and 5 will produce the correct sum but require much more time to apply than generalizations 1 to 3. The sixth generalization is only a little less effective than straight addition. Number 7 is a generalization in the sense that the way to get a sum is to add, but it fails to carry out the direction to find a method that is quicker than addition.

The three clues given affected the formulation of generalizations in a somewhat different manner. The first clue was effective chiefly with generalizations 1 and 6, the second with generalizations 1 and 2, and the third with generalizations 4 and 5. If generalizations 1, 2, and 3 are considered reasonably satisfactory, it is significant that only 17 of the 69 subjects were able to discover these without the help of one or more clues.

Problem 2 required the finding of a generalization covering the sum

TABLE 15
TYPES OF GENERALIZATIONS MADE FOR
PROBLEM 2 WITHOUT HELP OF CLUES

Type number	Frequency
1.....	9
2.....	4
3.....	4
4.....	6
5.....	7
6.....	0
7.....	15
8.....	22
9.....	1
Total.....	68

Generalizations 8 and 9 were correct.

of a series of odd numbers beginning with 1. The procedure varied slightly from that used with the first problem. No clues to the solution were given until the subject had formulated some kind of generalization, right or wrong. Consequently, in table 15 the first column shows the results obtained, without clues, by all in the group except one student, who omitted Problems 2 and 3.

Generalizations 8 and 9 were the correct ones for Problem 2. Twenty-three subjects made one or the other of these generalizations without the aid of clues, 28 more with the aid of the first clue, and 9 more with the help of the second clue. The remaining 8 subjects failed to make a correct generalization. The clues used with Problem 2 were as follows:

Clue Number 1: Look at the number in parentheses at the top of each column. This indicates the number of terms in each series.

Clue Number 2: Think of the number in parentheses at the top of each column in relation to the sum of each series.

The subjects used for this test were upper-division university students, who in respect to mental ability were without question a highly select group as compared with the general population. Sixty-eight of the 69 had taken the American Council on Education Psychological Examination (1949 edition). Their raw scores ranged from 61 to 169, with a median of 127. A raw score of 127 is at the seventy-ninth percentile based on a total of 31,121 freshmen from four-year colleges tested by the ACE. Yet only 23 of the 68 subjects were able to state a correct generalization without the help of clues. The mean ACE raw score for these 23 subjects was 128, compared with a mean score of 120

for the remaining members of the group. There was the usual amount of overlapping between the successful and the unsuccessful subjects. Evidently something in addition to mental ability, as measured by this test, accounts for the success of some and the failure of others.

The variety of generalizations proposed by the subjects throws some light on the mental processes used in discovering them. Samples drawn from Problem 1 (a series of consecutive numbers) are given below, ranging from direct statements of the generalization to awkward, involved verbal statements of varying degrees of value. Many of the subjects seemed to be aware of the proper generalization but were unable to verbalize it clearly. The lower-level generalizations are no improvement over conventional addition. The raw ACE Psychological Examination scores are given in parentheses after the number of each subject.

Subject 28 (133): "The mid-point of the series times the number of figures in the series will give the correct answer."

Subject 42 (152): "Take the middle number, or half-step between in case of even number of items, and multiply it by the number of items to be added."

Subject 38 (147): "Adding in multiples of 10, taking first and last numbers of each series and working toward center; then adding numbers not already included."

Subject 68 (137): "I used the multiple of ten or hundred usually computing from top and bottom into the middle; the exceptions were added last."

Subject 34 (99): "Find the mid-point, and add the pairs on either side of the mid-point. Then add the mid-point number."

Subject 67 (126): "Start with the first number, multiply that by the number of times it appears in the series and add the difference between the numbers."

Subject 57 (118): "Find the median. Make each other number in series equal this median number because the numbers coming before decrease at the same rate the numbers following increase. Add to get the sum."

Subject 51 (133): "By finding the one number in the column that seems to be the one which all the other numbers in the series can be made to agree with (by adding or subtracting with 2 numbers); then multiplying by the *number* in the series."

Subject 53 (128): "Take the middle number of the column, cause all numbers in the column to be the same as this number by process of addition and subtraction and multiply by the number of figures in the column. Same with even-numbered series, but *leave one* number off to be added later."

Subject 61 (144): "Find sum of first and last number. If series is even number of sums, divide this number by 2 and multiply this times the sum of the first and last number. If number of sums is uneven, divide again by two after subtracting one from the number of sums. Multiply the sum of first and last by this number, then add the number of the sum again."

Subject 26 (135): "I find the middle number in the series, count down from top and up from bottom, compute that number and then add most general number that I had found."

Subject 31 (106): "Upon examination it is found that there is a series of numbers in a column and there is an uneven amount of numbers in that series, then the middle or median number can be used as a central point and is half of the sum of each succeeding and preceding number. Therefore, the sum can be found by taking the median and adding this amount doubled as many numbers as above the median or below."

The two subjects with the highest and the lowest scores on the psychological examination gave the following generalizations. (Compared

TABLE 16
GENERALIZATIONS FOR PROBLEM 3 (PYRAMID) CLASSIFIED ACCORDING
TO GENERALIZATION MADE WITHOUT CLUES FOR PROBLEM 2

Generalization reached without clues for Problem 2	Generalization given for Problem 3					
	Not recorded	13	12	10*	11*	Total
1.....	0	4	0	3	2	9
2.....	0	2	0	2	0	4
3.....	0	1	0	1	2	4
4.....	0	3	1	2	0	6
5.....	0	2	0	4	1	7
6.....	0	0	0	0	0	0
7.....	0	8	0	6	1	15
8*.....	3	6	1	10	2	22
9*.....	0	0	0	1	0	1
Total.....	3	26	2	29	8	68

Asterisks indicate correct generalizations.

with the freshman norms these two students have percentile ranks of 99 and 4, respectively.)

Subject 33 (169): "Add top and bottom number, next to top and next to bottom, etc., which all equal the same answer. Multiply this number by the number of combinations found in the problem, then add any number which has not been used in a combination (the middle number)."

Subject 41 (61): "I found out the middle number and then multiplied by the number in the series."

From the data in tables 14 and 15, together with the sample generalizations in the foregoing quotations, several facts are apparent. First, the process of discovering a generalization is more difficult than is often assumed. This group of subjects is high in mental ability as compared with the general population; yet without the help of clues only 17 of the 69 subjects were able to state an acceptable generalization for Problem 1, and only 23 of 68 subjects for Problem 2. Second, the verbalization of a generalization is difficult. Even when the subject seemed

to have the idea in mind, the verbal statements were frequently awkward and were not reduced to essentials. The range of individual difference was wide. Third, and most important, there was little evidence of a technique for meeting a situation requiring generalization. Trial and error rather than logical analysis characterized the procedure of many of the subjects. In view of the claims of both secondary schools and colleges that "training to think" is one of their chief objectives, one may be justified in being disturbed about the outcome. Either learning how to discover generalizations is more difficult than is usually assumed, or the methods of teaching this kind of thinking are not too effective, at least for the group of subjects studied. The evidence from this group of subjects indicates that the ability to discover and state a generalization is not automatically guaranteed by high intelligence, but other factors are involved. Some of these factors might be revealed by an analysis of methods of teaching in respect to the relative emphasis placed on learning generalizations already formulated as compared with learning how to discover a generalization.

TRANSFER OF GENERALIZATIONS

Problem 3, consisting of a group of 81 x marks, arranged in the form of a pyramid, was used to test the application of the generalization covering the sum of a series of odd numbers. If the subject sensed that the horizontal rows of x marks constituted a series of odd numbers he could get the sum quickly by squaring 9, the number of x marks in the center column of the pyramid.

Table 16 shows the success of these students in making this transfer. Three subjects did not attempt Problem 3. Of the remaining 65, 29 gave generalization number 10, and 8 gave number 11, both of which are acceptable. However, the remaining 28 subjects gave generalization number 12 or 13, neither of which indicated transfer from the correct generalization used with Problem 2.

In working with Problem 2, 23 subjects formulated a correct generalization (number 8 or 9) without the help of clues. However, only 13 of the 23 gave a correct generalization on Problem 3, indicating a transfer of this knowledge. All 23 made the correct generalization on Problem 2, but 7 of them failed to see in Problem 3 a situation consisting of a series of odd numbers to which the generalization would apply. How to learn generalizations so that they can be transferred is one of the central problems of education at all levels. Although all but 8 of the 68 subjects eventually (with the help of clues) reached a correct generalization for Problem 2, there were only 23 who could make the necessary transfer of it to Problem 3.

CHAPTER IV. DEVELOPING A TESTING INSTRUMENT

Chapters iv and v will report an attempt to analyze the process of thinking during the solution of a particular problem. The primary purpose here is to describe as objectively as possible the various ways in which different individuals solve the same problem, and then to identify different patterns, if such exist, which characterize the thinking of sizable groups of individuals. In order to describe the thinking process in terms of patterns, it is necessary to set up certain categories into which different stages of the process may be classified. The dangers here are those encountered in any attempt at categorization. On the one hand, it is desirable to allow the greatest possible flexibility in describing the processes of thinking, in order to preserve the unique characteristics of individual procedures. On the other hand, the value of a system of classification resides in a disregard of nonessentials and a grouping of data around significant, essential factors. Some sacrifice has to be made in any attempt to describe patterns of thinking. The goal is to preserve as many of the individual variations as possible but at the same time group the patterns in such a way as to show the general direction followed.

In developing an instrument for use in analyzing problem-solving processes the basic principle followed was to derive the categories used in classifying from the actual thinking of individuals. This empirical procedure is at the opposite pole from a logically determined set of categories which might be set up in advance without reference to the processes actually used in individual cases.

The problem used in this section of the study is as follows:

A donkey and a mule were laden with wine. If 1 gallon of wine were taken from the donkey and placed on the back of the mule, the mule would have double the load of the donkey. But if 1 gallon were taken from the mule and placed on the donkey, the two animals would have equal burdens. How many gallons of wine did each animal carry at first?

STAGE I

The experimental work in developing an instrument for analyzing problem-solving thinking was done in three stages. First, in the fall of 1952, individual recordings were made of the solution of the problem by 68 upper-division college students. Each subject was encouraged to "think out loud" and to reveal as completely as possible the processes he employed in solving the problem. Each was told that his voice was being recorded. While the problem was being solved, the examiner

spoke only in response to questions or to ask "What are you thinking now?" after a subject had been silent for a rather long time. Occasionally it was necessary to encourage a student who was tempted to give up, by a comment such as, "You are doing very well." Questions pertaining to the solution of the problem, addressed to the examiner, were directed back to the student. For example, a common question was, "Does the problem mean that the gallon is taken from the donkey to the mule and then placed back again?" To this question the examiner typically answered, "Can you tell what it means from the way the problem is stated?" The experimenter made notes as the subject proceeded, and these, together with the transcription of the voice recordings, furnished the basic evidence of what each subject did. In the analysis of these materials, each different procedure was typed on a card and these cards were then compared and rearranged until a final set of 81 cards seemed to express all the individual procedures used by the subjects. These cards were then classified and given index numbers.

The 81 cards were organized into groups in the following manner. First there was a group of 3 cards giving different interpretations of the problem; then two cards allowing a choice of method of solution, either algebraic or logical reasoning. If the algebraic method was employed, there followed two groups of cards giving various equations and solutions. If the method of logical reasoning was chosen, there followed groups of cards giving choices of assumptions and various methods. There were then two groups of cards containing choices of solutions and methods of checking. This classified array of cards, representing the various processes employed by the first group of 68 subjects, was then tried with new subjects.

STAGE II

The second stage of the study was carried out with two groups; first, with 15 college students in the spring of 1953, and then, after some revisions, with 105 high school students. Some major changes in procedure were made during this stage.

The card-selection test employing the method used earlier by Lazerte was given to the 15 college students. After the students had read the problem, each was given the 81 cards, one group of cards at a time, and he was asked to select the card in each group which best represented his own solution of the problem. The cards selected were placed on the table in order from the one first selected to that containing the final answer to the problem. If a student finally selected the correct

solution, the test was terminated, but if he chose an incorrect one he was informed of his error and the procedure was repeated from the beginning. After this procedure had been used with some of the subjects, it became apparent that the test could not be administered satisfactorily in this manner. The student was faced with so many alternatives, some of which were difficult to understand and distinguish, that he became hopelessly confused. Only those subjects who chose the algebraic method and were able to solve the problem reported that the test was not overly burdensome.

In view of the difficulties just described, a second procedure was followed. The subject was instructed to begin by thinking aloud and was encouraged to think freely until he reached a phase in the process at which it was convenient to stop. The examiner then interrupted him and attempted to record his previous thinking by the selection of appropriate cards. In this the examiner and the subject worked together, the subject selecting the appropriate card from each group and the examiner explaining or clarifying the various alternatives as it appeared necessary. The index number of the card selected was then recorded and the procedure was continued. This procedure was repeated until the final solution was reached. Although this method proved superior to the one first employed, the subjects complained that the interruption of their thought processes made it difficult for them to resume their reasoning each time and thus resulted in a distortion of the problem-solving process. This procedure was therefore abandoned and a third method was tried.

The third method was to let the subject solve the problem in his own way, and to record his voice as he thought aloud. After the student had solved the problem to his satisfaction, he and the examiner attempted to select the cards which represented the procedure that had been followed. When the voice protocols were studied, certain omissions and inconsistencies in some of the card selections were evident. Subsequently, the card selections were modified to accord with the voice protocols, and each card was documented where possible by the numbered statements on the typed transcriptions of the voice recordings. Although the procedure introduced a degree of subjectivity into the selection of the cards, it resulted in a semiobjective record of the subject's problem-solving process which permitted a more satisfactory analysis than the voice recording alone, the method used in the first stage of the study.

After the revised method of using the cards was tried, the test was then given to 105 students in the 1953 summer session of the Oakland

Technical High School. The subjects, mostly juniors and seniors, were drawn from classes in the social studies and mathematics. They were sent individually to a small room where only the examiner and the subject would be present and where there would be no interruptions. Voice recordings were made and the cards were used as in the revised plan described in the preceding paragraph.

STAGE III

The third and final stage in the development of the testing instrument was carried out in the fall of 1953. In a further attempt to develop an objective test, the card-selection test, which had been expanded to 114 cards, was carefully screened and redesigned. The result was a selection of 38 cards which included the essential steps and procedures of the previous test. As in the first test, the cards were arranged in groups and each group was placed in separate envelopes, so that a completely objective procedure similar to Lazerte's could be followed. This development of the test appeared to satisfy both of the requirements that had previously been set. It allowed for freedom of thought processes because it presented the student with a wide variety of possible ways, empirically determined, to solve the problem, and the test was simple to take, resembling a multiple-choice design with which most subjects were familiar.

In this stage of the study, 61 college subjects were tested individually, with the revised set of cards. The cards were supplemented with voice recordings. After the examiner had explained the purpose of the test, the following directions were read aloud.

There are many ways to solve this problem. We have a number of different ways broken down step by step, with each step printed on a card. You are to select from the steps the procedure which will lead to the solution of the problem, making as few steps as possible. Some of the steps will lead quickly to the solution, and others will not lead to the solution at all.

When you think you have found the correct solution, you will be required to verify it. If your solution does not check, you will have to begin again. If you decide that you are on the wrong track before you reach a solution, you may stop and begin again.

Whenever you begin again, you will have to start from the very beginning.

I will tell you specifically what to do as we go along. Here is the problem. Read it carefully and tell me when you are ready to begin solving it.

At this point the problem, typed on a card, was handed to the student to read. When the student indicated that he was ready to solve the problem, the test procedure was started with the following alternatives:

"There are two general approaches to the solution of this problem:

(1) using algebra, and (2) using logical reasoning. Which way do you wish to try first?"

If a subject elected to solve the problem algebraically, he was then told, "People have attempted to solve the problem by setting up equations in one unknown and in two unknowns. Which way do you wish to try to solve it?" Or, "Which way do you think is correct?" Depending on the student's choice, he was given a group of cards on which were typed a number of different equations which had been most frequently used by subjects previously tested. In all, there were 11 cards, 4 in the group of equations with one unknown and 7 in the group of equations with two unknowns. Only one set of equations was correct in the whole group of 11 cards. When the student selected a set of equations, the examiner said, "We are presuming that you could solve these equations. If you were to solve them correctly, you would get the following answer." The solutions to the equations, which were printed on the back of each card, were read to the student directly, and he was asked to verify the given solution. If the solution did not check with the answer, he was required to begin again.

If a student chose to solve the problem through logical reasoning, he was instructed as follows:

This is like a multiple-choice test. I have here five envelopes in which there are a number of cards with instructions typed on them. Your task is to go through these envelopes, one at a time, and select the one card from each that you think is best or most correct. In the end you will be given a number of possible solutions from which you must choose the correct solution. Then you will have to verify your solution, and if it does not check, you will have to begin again. There are several possible ways of arriving at the correct solution, so expect to find more than one correct choice in each envelope. Also expect to find some incorrect choices. Here are the cards in the first envelope. Select the one you think is best.

There were six cards in the first envelope. These were spread on the table in front of the student. When the student selected a card, its number was recorded, and the remaining cards were replaced in the envelope. Then cards from the second envelope were given to the student, and the procedure was repeated. Each card selected was placed on the table along with the problem.

To some extent, what the student selected in the first envelope determined which alternatives he would have to choose from subsequently. For example, if a student chose Card 2 from the first group, he was given only two cards from the second envelope, Cards 10 and 13. Card 2 states, "There is only one animal carrying a load, which is 1 gallon." Card 10 states, "Start with 1 gallon and work out the other number.

The answer will have to be 1 gallon and some other number of gallons." Card 13 states, "Select any combination which seems to satisfy one part of the problem and test to see if it will satisfy all of the problem. Continue this process until the solution is found." Each student was required to choose from among all the alternatives in each of the subsequent envelopes, regardless of his previous selections.

The one exception in the test procedure described above resulted when a student selected Card 1 from the first envelope. This card states, "There can be only one gallon on each animal to begin with if the mule's load is doubled after moving one gallon from the donkey to the mule." If a student selected this card, he was informed that the solution followed directly from his choice and did not involve further reasoning. The solution, 1 gallon on each animal, is implied in the statement. In the verification of this solution, the student generally revealed that he had misinterpreted the problem, a mistake made by a large number of both high school and college students. If a student who selected Card 1 did not recognize his own mistake when asked to verify the solution, the examiner explained it to him, and the test was started again from the beginning.

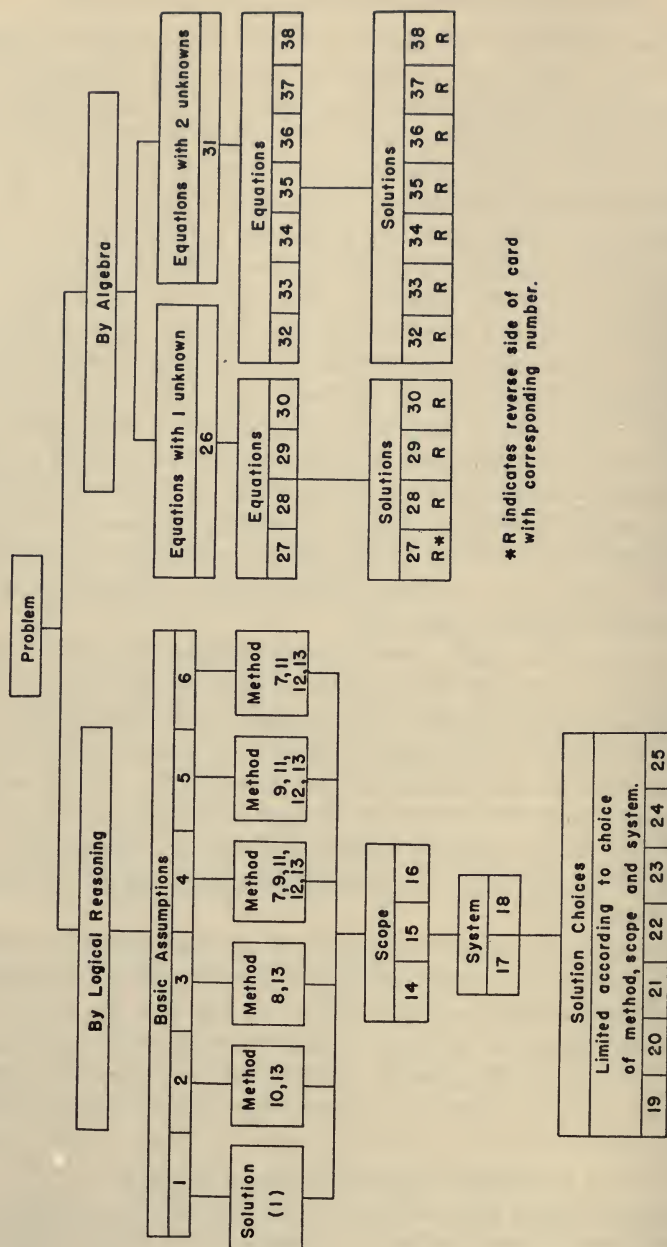
The examiner then read to the student a number of possible solutions, depending on the cards the student had selected, and instructed him to select the correct solution, if it was among those presented. After reading a solution, the examiner waited for the student's response before reading the next. When a student selected a possible solution, the examiner said, "Now show me how your solution solves the problem." If the solution proved correct, the test was terminated. If the solution proved incorrect, then the entire test procedure was repeated from the beginning.

The major objective of the study should be kept in mind, namely, to find whether there are some patterns of thinking that are common to a considerable number of individuals, or whether the choices seem random and devoid of pattern. Whatever patterns exist could be described in terms of the general structure of the test instrument, the nature of which is indicated in figure 9 and in the content of the test cards.

SYSTEM OF CARDS TO ACCOMPANY FIGURE 9

Problem: A donkey and a mule were laden with wine. If one gallon of wine were taken from the donkey and placed on the back of the mule, the mule would have double the load of the donkey. But if one gallon were taken from the mule and placed on the donkey, the two animals would have equal burdens. How many gallons of wine did each animal carry at first?

STRUCTURE OF TESTING INSTRUMENT



*R indicates reverse side of card with corresponding number.

Fig. 9. Structure of Testing Instrument.

Numbers in the rectangles correspond to the card serial numbers shown on the following pages.

I. By logical reasoning

A. Basic assumptions (Envelope 1)

- 1 (1).⁹There can be only one gallon on each animal to begin with if the mule's load is doubled after moving one gallon from the donkey to the mule.
- 2 (2). There is only one animal carrying a load, which is one gallon.
- 3 (3). The loads are equal to begin with.
- 4 (4). The loads are not equal to begin with, but I am not certain of the exact difference between the loads.
- 5 (5). There must be a difference of one gallon between the loads.
- 6 (6). There must be a difference of 2 gallons between the loads.

B. Method (Envelope 2)

- 1 (7). Select combinations of numbers having a difference of two. Test these combinations to see which will satisfy the conditions of the problem.
- 2 (8). Select pairs of equal numbers such as 1-1, 5-5, 3-3, and test these combinations to see which will satisfy the conditions of the problem.
- 3 (9). Select combinations of numbers having a difference of one. Test these combinations until the solution is found.
- 4 (10). Start with one gallon and work out the other number. The answer will have to be one gallon and some other number of gallons.
- 5 (11). Select any two equal numbers such as 1-1, 5-5, 3-3, and work backward to find what the original load would have been. Test each number combination thus obtained to see if they satisfy the conditions of the problem.
- 6 (12). Select numbers in the ratio of 1 : 2 and work backward to find what the original load would have been. Test combinations of numbers thus obtained until the correct solution is found.
- 7 (13). Select any combination which seems to satisfy one part of the problem and test to see if it will satisfy all of the problem. Continue this process until the solution is found.

C. Scope (Envelope 3)

- 1 (14). The original loads will be less than 5 gallons.
- 2 (15). The original loads will be less than 10 gallons.
- 3 (16). The original loads may involve numbers greater than ten.

D. System (Envelope 4)

- 1 (17). Start with the lowest combination of numbers and systematically test larger combinations until the solution is found.
- 2 (18). Select only those combinations of numbers which seem most likely to solve the problem. Do not select the number combinations in any particular order.

E. Solution Choices (Envelope 5)

(19). Following method of Card 7:

(a)	(b)	(c)
0-2	4-6	9-11
1-3	5-7	10-12
2-4	6-8	11-13
3-5	7-9	12-14
	8-10	13-15

⁹ Numbers in parentheses are the card serial numbers.

(20). Following method of Card 8:

(a)	(b)	(c)
1-1	6-6	11-11
2-2	7-7	12-12
3-3	8-8	13-13
4-4	9-9	14-14
5-5	10-10	15-15

(21). Following method of Card 9:

(a)	(b)	(c)
0-1	5-6	10-11
1-2	6-7	11-12
2-3	7-8	12-13
3-4	8-9	13-14
4-5	9-10	14-15

(22). Following method of Card 10:

(a)	(b)	(c)
1-1	1-6	1-11
1-2	1-7	1-12
1-3	1-8	1-13
1-4	1-9	1-14
1-5	1-10	1-15

(23). Following method of Card 11:

(a)	(b)	(c)
Equal-Original	Equal-Original	Equal-Original
1-1	0-2	5-5
2-2	1-3	4-6
3-3	2-4	5-7
4-4	3-5	6-8
	8-8	7-9
	9-9	8-10
		10-10
		11-11
		12-12
		13-13
		14-14
		15-15

(24). Following method of Card 12:

(a)	(b)	(c)
1:2 - Original	1:2 - Original	1:2 - Original
1-2	2-1	4-8
2-4	3-3	5-10
3-6	4-5	6-9
		7-14
		8-16
		9-11
		10-12
		11-13
		12-14
		13-15

(25). Following method of Card 13:

(a)	(b)	(c)
0-2	4-6	7-11
1-1	5-7	8-10
1-3	6-8	9-11
2-1	6-9	10-12
2-4	7-9	11-13
3-3		12-14
3-5		13-15
4-5		14-16

II. By Algebra

A (26). Equations with one unknown (Envelope 6)

- 1 (27). $x - 1 = 2x$
 $2x - 1 = 2x - 1$
- 2 (28). $x - 1 = 2(x + 1)$
 $2(x + 1) - 1 = x + 1$
- 3 (29). $2(x - 1) = x + 1$
 $x + 1 = x - 1$
- 4 (30). $2(x - 1) = x$
 $x + 1 = x - 1$

B (31). Equations with two unknowns (Envelope 7)

- 1 (32). $d - 1 = 2(m + 1)$
 $m - 1 = d + 1$
- 2 (33). $m + 1 = 2d$
 $m - 1 = d + 1$
- 3 (34). $m - 1 = d$
 $d - 1 = 2m$
- 4 (35). $2(d - 1) = m + 1$
 $m - 1 = d + 1$
- 5 (36). $m - 1 = d$
 $2(d - 1) = m$
- 6 (37). $2(d - 1) = m$
 $d + 1 = m - 1$
- 7 (38). $d - 1 = 2m + 1$
 $m - 1 = d + 1$

C. Solutions (on reverse of cards in Envelopes 6 and 7)
(Envelope 6)

- 1 (27-R). $x = -1$
- 2 (28-R). $x = -3$
 $x = 0$
- 3 (29-R). $x = 3$

- 4 (30-R). $x = 2$
- (Envelope 7)

- 5 (32-R). $m = -5$
 $d = -7$
- 6 (33-R). $d = 1$
 $m = 1$
- 7 (34-R). $m = -2$
 $d = -3$
- 8 (35-R). $d = 5$
 $m = 7$
- 9 (36-R). $d = 3$
 $m = 4$
- 10 (37-R). $d = 4$
 $m = 6$
- 11 (38-R). $d = -6$
 $m = -4$

CHAPTER V. ANALYSIS OF DATA

This chapter deals with the data derived from the three-stage experiment that was described in chapter iv. The reader should keep in mind that in stage one of the experiment the subjects solved the problem through a completely unstructured and uncontrolled process of thinking. In stage two there was a partial control through the use of verification of the thinking process by the selection of appropriate cards *after* the subject had attempted to solve the problem, and in stage three there was a more direct structuring of the thinking process in which the sequence of thinking was expressed by the selection of appropriate cards from the group of 38 possibilities *during* the process of solution. The principal data for the experiment are based on this third stage of the experiment; but before the analysis of these data is presented a few samples of problem solving selected from the first stage of unstructured thinking will be shown.

SAMPLE CASES

Five cases will be described to illustrate varieties of thinking. These include the tests of the two persons making the highest and the lowest score on the American Council on Education Psychological Examination, and three with intermediate scores. The samples illustrate the lack of systematic procedure in attacking problems, and some of them show clearly the periods of confusion which many subjects experienced in their attempt to solve the problem.

Sample 1.—Subject 33 was the college student that had the highest score on the psychological examination, namely a raw score of 169. After reading the problem, she formulated it simply and clearly, thinking in terms of shifting one gallon one way or the other. A shift one way results in the load on one animal being twice that on the other. A shift in the opposite way results in the loads being equal. The student represented the conditions graphically on a piece of paper. Starting with one, she tried successive numbers, attempting to determine what the original loads would have to be to satisfy the “double-load” condition, then testing to see if the same numbers would satisfy the “equal-load” condition. Subsequently, she focused upon the double-load condition, saying that the results “after the transfer” would be numbers in the ratio of 1 to 2. Writing a series of such number combinations, she determined in an adjacent column the number of combinations representing the original loads by working backward. Then she asked herself, “Now, would any of those when worked the opposite way be equal?” By testing each combination in the original load column, she

quickly found the numbers 5 and 7. However, for some reason, she failed to recognize 5 and 7 as the answer and seemed to become temporarily confused. Starting again from the beginning, she very quickly clarified the problem and reviewed her previous work. This time she selected the combination 5 and 7, saying, "This one should work." She tested it separately in terms of her original model, expressing it thus: "4-8 double; 6-6 equal." After finishing her work, she said that what had confused her was that she had failed to subtract the 1 gallon from the 5 when it was transferred to the mule. Thus the numbers were not in the ratio of 1 to 2 after the transfer. This one error had confused her so that she had to reclarify the problem before she saw that it was the solution. Her total time was 8 minutes.

Sample 2.—Subject 41 made the lowest score in the group on the ACE intelligence test—a raw score of 61. She formulated the problem incorrectly but promptly gave the solution as one gallon on each animal. In explaining her solution, she said that she took 1 gallon away and then returned it. When she took the gallon away, there were 2 gallons on one animal and 0 gallons on the other. She reasoned that 2 was double 0 which, it may be observed, was an error made by a good many of the subjects, particularly those at the high school level. Even after the examiner directed her attention to her error and suggested the correct way of interpreting the problem, she was unable to change her previous mental set and reformulate the problem. She said, "I guess I have that too firmly in mind." Although she used only 45 seconds, her 1-1 answer was firmly fixed and she would make no further attempts.

Sample 3.—Subject 36 made a raw score of 106 on the psychological examination. He began by expressing the two conditions of the problem in the form of algebraic equations with two unknowns. Although the equations were algebraically correct, he said he did not intend to solve the problem algebraically but that the equations were merely to clarify the problem to him and serve as a guide. He then proceeded to try to find a number which when increased by one would equal two times another number minus one. That is, to find x and y in the equation $y + 1 = 2(x - 1)$. His plan was to "fool around with numbers" until he found the correct ones. However, he first said that y would be odd and x would be even, for reasons he did not clearly explain. Then he started with 10, because it was "convenient," and assuming both sides of the equation to equal 10 he determined what the unknowns were, arriving at $y = 9$ and $x = 6$. Then substituting these numbers in the second equation, he found that this was not the solution. Then he decided to reverse

the process, starting with the second equation instead of the first, namely, $y - 1 = x + 1$. Before he did this, however, the thought occurred to him that the sums of unknowns should be equal, and from this he tried to make some connection to relate the two equations. Failing in this, he returned to reread the problem and review his thinking. He reclarified in his mind the fact that he was trying to solve for x and y . Then he tried reversing the process he originally used and started with the second equation. He assumed that each side of the second equation equalled 8, solved for x and y , and substituted the result in the first equation. After this failed, he again read the problem, saying, "Maybe I haven't read it right." He considered that he was approaching the problem in the wrong way, thinking the method he first tried had failed. He expressed discouragement and inability to think of another "approach." Finally, he thought he would just try numbers and see whether by fitting them into the equation he could also find the other number and then would keep trying different numbers until he hit upon the correct combination. He decided to start with odd numbers. After trying combinations beginning with 1 through 5, he stopped to check over his work before continuing. Finally he started with $y = 7$ and solved for $x = 5$, which satisfied the conditions of the problem. He said that the first and the last systems were actually identical, except that in the first he had failed to select numbers systematically to try in his equations. After making sure his equations were right, he felt certain that by trying numbers systematically he would eventually find the solution. His total time was 20 minutes, 30 seconds.

Sample 4.—Subject 10 was a college student with a psychological-test raw score of 130. She formulated the problem correctly and then began choosing pairs of numbers at random. She asked the examiner if both animals started with equal loads and then realized that this was part of the problem. Focusing upon the fact that in one condition the one animal would have to have double the load of the other, she began dealing with such numbers as 2-1, 4-2, and 8-4, working back to what the load was originally by reversing the transfer of the gallon of wine. She then stated that any of the number combinations thus determined would work. After the examiner reminded her that there was more to the problem, she used the same procedures and then tested each combination to see which would also satisfy the second condition. In this manner she found the correct answer. After determining the correct combination, she was faced with so many numbers on her page that when asked to state the answer she became momentarily confused and gave an incorrect pair of numbers. The examiner asked her to clarify

her answer. She reacted by expressing doubt of her understanding of the problem and of her ability to recall how she had found the answer. However, she quickly recovered from this confusion and gave the correct answer. Her total time was 11 minutes.

Sample 5.—Subject 7 was a college student with a raw score of 123 on the psychological test. She formulated the problem correctly and made the assumption that each animal would have at least one gallon to start with and that the loads would be small numbers. She proceeded to solve by focusing her attention entirely on the transfer of loads. On a sheet of paper her beginning attempts were in the form of tally marks, each mark representing one gallon. Starting with one gallon on each animal, she added tally marks and then crossed out as was appropriate in actually transferring the gallon of wine. This was the only system she used. Otherwise the process seemed to be random trial and error of number combinations of increasing size. There was even repetition of the same numbers. However, she eventually came upon the correct answer, using a total time of 19 minutes.

Records such as the foregoing five samples were secured from 68 college students in stage one of the experiment. There was no attempt to treat these records quantitatively; rather, the purpose was to secure from them as great a variety as possible of procedures which, taken together might constitute a preliminary catalogue of operations employed in attempting to solve the problem. It was from records such as these that the group of cards used in the second stage of the experiment with 105 high school pupils was derived. The following sample is transcribed directly from the voice record of one of the high school students.

Subject: "Can't the donkey and the mule have more than 1 gallon?... It doesn't specifically say.... You're not given a figure to begin with... If you take 1 gallon off the donkey, and place it on the back of the mule, that doesn't tell you how many gallons were on the donkey at first... does it? And if you take the extra gallon off the mule and place it on the donkey, they'll be equal again... so I conclude from here... I couldn't give you any numbers...."

Experimenter: "Can you think of any numbers that would do?"

Subject: "Well, that's just a guess.... Say the donkey and the mule had 2 gallons each. You assume that.... If 1 gallon were taken off the donkey and placed on the back of the mule, the mule would have 3 and the donkey would have 1.... They're unequal. If 1 gallon were taken off the mule and placed on the donkey, they'd both have 2—what they started out with.... It says here: 'How many gallons of wine did each animal carry at first?'... I can't get it... without given any figures at the beginning. Can it be solved? By algebra? Oh! I didn't read it... I didn't read 'have double the load.' I didn't take that into my mind. So... both had... 1 gallon to begin with."

Experimenter: "How did you figure that?"

Subject: "Take 1 off the donkey, and put it on the mule... then the mule would

have double its load. . . . It would have 1 gallon at first, and then it would have 2 after it had received the 1 from the mule. . . . So . . . you subtract that from 2; 1 from 2 is 1, so that equals back to 1."

When the student had reached this point in his thinking, the examiner said, "Now, let's look at the cards," and he spread on a table the cards representing the different steps used in solving the problem. Discussing the cards, they together selected the series which most appropriately represented the kind and sequence of thinking the student had followed.

DATA FROM TEST 6, COLLEGE GROUP

When all the high school subjects had finished, the catalogue of operations employed had increased to a total of 114 cards. In the process of analyzing the statements on these cards and attempting to discover patterns of thinking from them, it became apparent that some grouping of minor variations was necessary if patterns were to be discerned. For example, an analysis of the sequence of operations for each of the 105 subjects revealed facts such as the following. After reading the problem, 61 subjects began their solution with the operation described on Card 2, 21 subjects began with Card 1, and 10 subjects with Card 5; the remaining 13 subjects showed little agreement in their choices. Of the 61 who began as described on Card 2, all agreed on the second step, namely, Card 4. In the third step, however, only 23 of the 61 subjects agreed on any one card; and when a solution was reached in the fifth step, the choices of only 18 of the 61 subjects who started in the same manner coincided. The second-highest amount of agreement was among a group of 21 subjects who began with Card 1 as their first step, but no two of the 21 followed the same pattern to the final step. Of the 105 subjects on whom data were available, the largest number agreeing on any one pattern from start to finish was 18.

The principal objectives of the second stage of the study with the high school student subjects were (a) to identify as many variations in the problem-solving process as possible and then (b) to use the larger catalogue of operations with the subjects in stage three. However, in the attempt to have these subjects select from the total of 114 cards those containing statements that corresponded to their individual thinking process, it was found that they were confused by the large number of minor alternatives presented and that 114 cards were too many. Therefore, the operations on the cards were reexamined to see whether a major condensation was possible. From this reclassification

TABLE 17

SERIAL ORDER OF PROCESSES USED IN SOLVING THE DONKEY-MULE PROBLEM
SIXTY-ONE COLLEGE STUDENTS

Maximum number of trials needed for solution	Serial order of processes	Number of cases per pattern	Total cases per trial	Abandoned without final solution
1	L (logical).....	10	..	..
	A (algebraic).....	1	11	..
2	LL.....	17	..	2
	AA.....	5	..	1
	AL.....	4	26	..
3	LLL.....	6	..	..
	LAL.....	1	..	..
	LAA.....	1	..	..
	ALL.....	1	..	..
	AAL.....	1	10	..
4	LLLL.....	1	..	..
	AAAA.....	1	..	..
	LAAL.....	2	..	..
	LAAA.....	1	..	..
	AAAL.....	1	..	..
	AALA.....	1	..	..
	ALAA.....	1	..	..
	ALLL.....	2	10	1
5	LLLAL.....	1	..	..
	AAALL.....	1	2	..
6	LLAALL.....	1	..	..
	AAALLL.....	1	2	..
	Total.....	61	61	4

came the final sequence of 38 cards listed at the end of chapter iv. This set of cards was then used in testing 61 college students in a final attempt to identify and describe patterns of thinking. The data from this group of subjects provide the basis of this chapter.

The reader should keep in mind that the different problem-solving operations are designated in this chapter by the serial numbers given to each operation described at the end of chapter iv, that the cards numbered 1 to 25 are for solutions by logical reasoning, and the cards numbered 26 to 38 are for solutions based on the special type of logical

TABLE 18

FREQUENCY OF SOLUTION PATTERNS IN SOLVING THE DONKEY-MULE PROBLEM

Section A. Logical patterns		Trials						Total
Serial no.	Card numbers	1	2	3	4	5	6	
1	1 abandoned.....	7	3	..	1	..	..	11
2	2-10-14-18.....	1	..	..	..	..	..	1
3	2-10-15-17-(abandoned)....	..	..	..	1	..	..	1
4	3- 8-14-17.....	2	..	..	..	..	..	2
5	3-13-15-17.....	1	..	..	..	..	..	1
6	4- 7-14-18.....	1	..	..	..	..	..	1
7	4- 7-15-17.....	..	1	1	..	..	..	2
8	4- 7-16-17.....	..	..	..	1	..	..	1
9	4- 7-16-18.....	..	2	..	..	..	..	2
10	4- 9-14-17.....	1	1	..	..	..	..	2
11	4- 9-14-18.....	..	..	1	..	..	..	1
12	4- 9-15-17.....	..	..	1	..	..	..	1
13	4-11-14-17.....	..	1	..	1	..	..	2
14	4-11-14-18.....	1	..	..	..	..	..	1
15	4-11-15-17.....	..	..	..	..	1	..	1
16	4-12-14-17.....	3	2	..	..	1	..	6
17	4-12-14-18.....	2	1	..	..	..	..	3
18	4-12-15-17.....	2	1	..	..	..	1	4
19	4-12-15-18.....	..	..	1	..	..	..	1
20	4-12-16-17.....	..	1	2	..	..	..	3
21	4-13-14-17.....	2	4	..	..	..	..	6
22	4-13-14-18.....	2	..	..	..	..	..	2
23	4-13-15-17.....	1	5	2	..	1	..	9
24	4-13-16-17.....	3	..	2	1	..	..	6
25	4-13-16-18.....	..	1	1	..	..	..	2
26	5 abandoned.....	1	..	..	..	..	..	1
27	5- 9-14-17.....	1	2	..	..	..	..	3
28	5- 9-15-17.....	2	..	1	..	..	..	3
29	5- 9-15-18.....	1	..	..	..	1	..	2
30	5-12-16-17.....	..	1	..	..	..	..	1
31	5-13-15-17.....	..	1	..	..	..	..	1
32	5-13-15-18.....	..	1	..	..	..	..	1
33	5-13-16-17.....	..	1	..	..	..	..	1
34	6 abandoned.....	1	..	..	..	..	..	1
35	6- 7-14-17.....	1	..	1	..	..	..	2
36	6- 7-15-17.....	2	3	..	3	..	1	9
37	6- 7-15-18.....	1	1	..	..	..	..	2
38	6-12-14-17.....	1	..	..	..	..	..	1
39	6-13-14-17.....	..	..	1	..	..	..	1
40	6-13-15-17.....	1	1	..	..	..	..	2
Total (logical patterns)		41	34	14	8	4	2	103

TABLE 18—Continued

Section B. Algebraic patterns		Trials						Total
Serial no.	Card numbers	1	2	3	4	5	6	
	Attempted but abandoned without selecting cards (no pattern).....	6	2	..	1	..	..	9
41	26-27.....	2	1	2	..	..	..	5
42	26-28.....	1	2	1	..	..	..	4
43	26-29.....	2	..	..	..	..	..	2
44	26-30.....	..	..	1	..	..	..	1
45	31-32.....	1	2	..	..	..	..	3
46	31-33.....	1	2	1	..	..	..	4
47	31-34.....	3	2	1	1	..	..	7
48	31-35.....	1	3	1	4	..	..	9
49	31-36.....	1	..	..	..	..	..	1
50	31-37.....	2	1	2	..	..	..	5
51	31-38.....	..	1	1	..	..	..	2
	Total (algebraic patterns).....	20	16	10	6	..	..	52
	Grand total (both patterns).....	61	50	24	14	4	2	155

reasoning that employs algebraic methods. In designating the two categories as "logical" and "algebraic," there is of course no implication that algebraic methods are not logical.

Table 17 gives an over-all view of the efforts of the 61 subjects in solving the problem. Eleven subjects solved it on the first trial, 10 of them using logical reasoning and 1 using algebra. Twenty-six subjects made two trials in attempting to solve the problem, 23 getting the correct solution and 3 giving up without solving it. Of those making two trials, 17 used logical reasoning in both trials, 5 used algebraic methods in both trials, and 4 used algebra in the first trial but changed to logical analysis in the second. As indicated in table 17, 2 subjects needed six trials to reach a solution of the problem. One of these subjects used logical reasoning in the first two trials and algebraic methods in the next two and then returned to logical analysis for the last two trials. The other subject tried algebra in the first three trials and then changed to logical reasoning for the last three. As indicated in the table, 41 subjects consistently used only one or the other of the two methods until they had solved the problem, and 20 shifted from logical to algebraic methods, or the reverse. Four of the 61 subjects never succeeded

in solving the problem, 3 abandoning it after two trials and 1 after four trials.

Fifty of the subjects used logical reasoning as their final method of solving the problem, and 34 of the 50 used only logical analysis for all their attempts. Eleven subjects employed algebraic methods in their final solution, but only 7 used algebraic methods for all trials. The reason for not using algebra as a means of solving a problem which is so obviously suited to this method is difficult to explain, unless it is that the students felt insecure in employing algebraic methods.

Table 17 showed the variations among 61 subjects in the use of either logical or algebraic methods of solving the problem. Table 18 goes further and shows the variations in the individual patterns of thinking. Subjects following the logical method are shown in Section A of the table, and those following the algebraic method in Section B. The numbers in the patterns indicate the successive cards selected by the subjects in solving the problem. The operation represented by each number can be identified by reference to the serial numbers of the 38 cards given at the end of chapter iv. The 61 subjects made a total of 155 trials in solving the problem. Nine of these 155 trials were made by subjects who started to use algebra but were unable to decide even on the alternative of using equations with one or two unknowns, and who quickly abandoned the algebraic solution without selecting cards to indicate a pattern of thinking. The remaining 146 trials were carried through by the selection of cards that described the method of solution employed; 51 different patterns of thinking were represented in these 146 trials. The 51 patterns and the number of subjects using each are shown in table 18.

Table 18 should be read as follows: The first pattern of solution consisted in selecting Card 1, which read, "There can be only one gallon on each animal to begin with if the mule's load is doubled after moving one gallon from the donkey to the mule." Since this statement involves an incorrect solution to the problem, the next step must necessarily be to abandon it, reread the problem, and begin a second trial. As indicated in the table, 7 subjects followed this pattern of thinking for their first trial, 3 subjects on their second trial, and 1 subject on his fourth trial.

The second pattern of thinking shown in table 18 followed the card sequence 2-10-14-18, only one subject showing this pattern. Expressed as stated on the cards, this subject's pattern of reasoning was as follows:

<i>Card 2—Basic Assumption:</i>	There is only one animal carrying a load, which is one gallon.
<i>Card 10—Method:</i>	Start with one gallon and work out the other number. The answer will have to be one gallon and some other number of gallons.
<i>Card 14—Scope:</i>	The original loads will be less than 5 gallons.
<i>Card 18—System:</i>	Select only those combinations of numbers which seem most likely to solve the problem. Do not select the number combinations in any particular order.
<i>Solution Choice:</i>	Since Card 10 leads to Solution Card 22, and since Card 14 limits the choice to loads under 5 gallons, the first column on Card 22 lists the only possible combinations, none of which is the correct solution. Consequently this subject must begin a second trial.

Nine subjects followed pattern 36 in table 18. The reasoning of this group is indicated in the card series 6-7-15-17. The statements on these cards read as follows:

<i>Card 6—Basic Assumption:</i>	There must be a difference of two gallons between the loads.
<i>Card 7—Method:</i>	Select combinations of numbers having a difference of 2. Test these combinations to see which will satisfy the conditions of the problem.
<i>Card 15—Scope:</i>	The original loads will be less than 10 gallons.
<i>Card 17—System:</i>	Start with the lowest combination of numbers and systematically test larger combinations until the solution is found.
<i>Solution Choice:</i>	Since Card 7 leads to Solution Card 19, and since Card 15 limits the solution to numbers under 10, only the first two columns of Card 7 are appropriate and the correct answer is the sixth combination tested.

The sequence of thinking for each of the 51 patterns shown in table 18 may be seen by reading the cards listed successively in each pattern. Aside from the first pattern, in which 11 subjects began with the reasoning indicated in Card 1, there were 4 patterns followed by 9 subjects each, 1 pattern by 7 subjects, and 3 patterns by 6 subjects each. No other pattern of thinking was followed by more than 5 subjects, and there were 20 patterns followed by only 1 subject. The problem-solving thinking of these 61 subjects exhibits far more variety than uniformity, and this is true in spite of the fact that the minor variations represented by 114 cards in stage two of the study were reduced to 38 more general categories in the present stage.

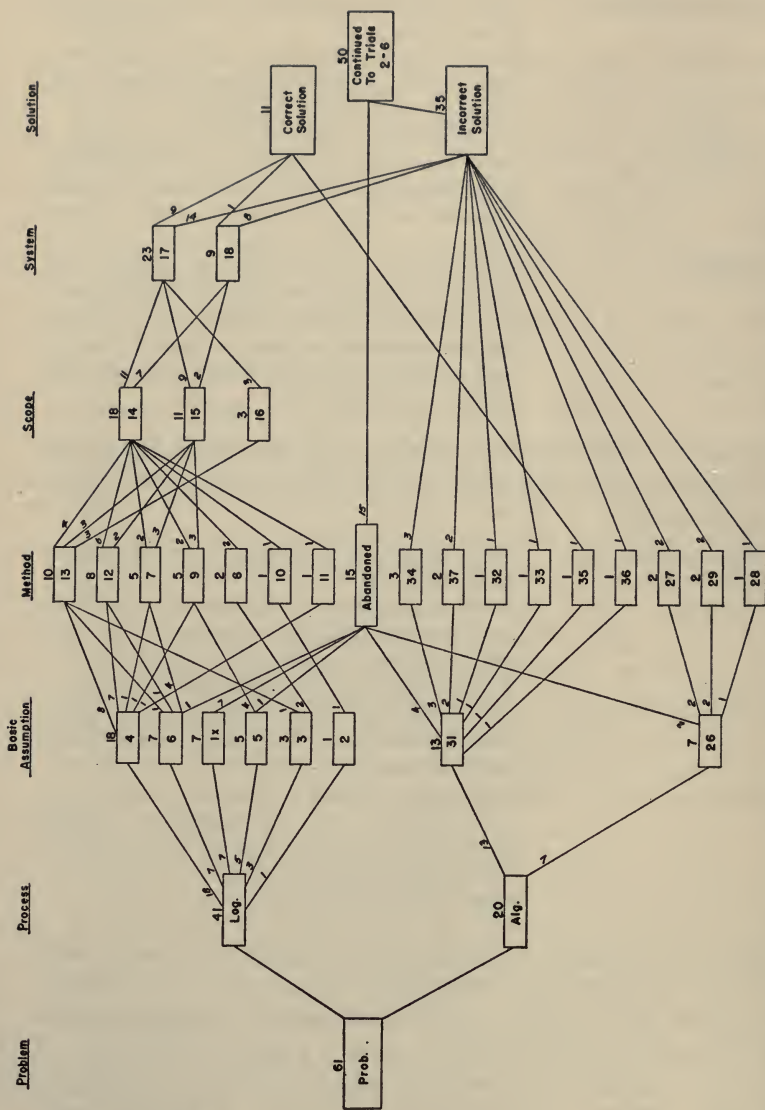


Fig. 10. Patterns of problem-solving thinking of 61 college students on the donkey-mule problem, first trial.
 Number of subjects above rectangles. Operations within rectangles.
 The students indicated by the X misinterpreted the problem. The problem was explained, and they began again.

The variations in patterns of thinking followed in solving the problem used in this study are shown more effectively in figs. 10 to 15. In these charts the variations for the entire group of 61 subjects are shown for each of the trials in solving the problem. The structure of the six charts is similar, and will be explained in reference to the first one.

Figure 10 shows the patterns followed by 61 subjects in the first trial. The numbers within the rectangles refer to the solution cards listed at the end of chapter iv. The numbers above the rectangles give the number of subjects using the operation described on the corresponding card. The major subdivisions of the thinking process are indicated by the terms across the top of the chart, proceeding from problem to process, basic assumption, method, scope, system, and solution. Figure 10 should be read as follows: Sixty-one subjects attempted to solve the problem, 41 by using logical reasoning and 20 by using algebraic processes. Of the 41 employing logical reasoning, 18 chose Basic Assumption Card 4, 7 chose Card 6, and so forth. Of the 18 who chose Basic Assumption Card 4, 8 chose Method Card 13, 7 chose Card 12, and so forth. By following the lines, one may find the number of subjects choosing each of the remaining cards. Eleven of the 61 succeeded in solving the problem on the first trial. Of the remaining 50 subjects, 35 arrived at an incorrect solution, and 15 gave up during the process without reaching any solution. These 50 subjects then proceeded to trial 2. As shown in figure 10, 11 subjects were able to solve the problem on their first trial. Ten of the 11 solved it by logical reasoning, this being 24 per cent of the 41 subjects using that method. However, of the 20 subjects who tried to solve the problem by using algebra, only 1, or 5 per cent, of the group arrived at the correct answer. Seven of the 20 tried to solve the problem by using equations with only one unknown.

As shown in figure 11, 23 subjects reached a correct solution on their second trial. Of the 34 who used logical reasoning, 19, or 56 per cent, reached a correct solution, but only 4, or 25 per cent, of the 16 who tried algebraic methods succeeded.

Twenty-four subjects went on to a third trial, as indicated in figure 12. Of the 14 who used logical reasoning, 9, or 64 per cent, reached a correct solution, but only 1 of the 10 using algebraic methods obtained the correct answer. Fourteen of this group continued on to a fourth trial, 4 to a fifth, and 2 to a sixth trial, as shown in figures 13-15.

By using from one to six trials, 57 of the 61 subjects eventually solved the problem. Four subjects gave up without reaching a solution, 3 after the second trial and 1 after the fourth. Although figures 10 to 15

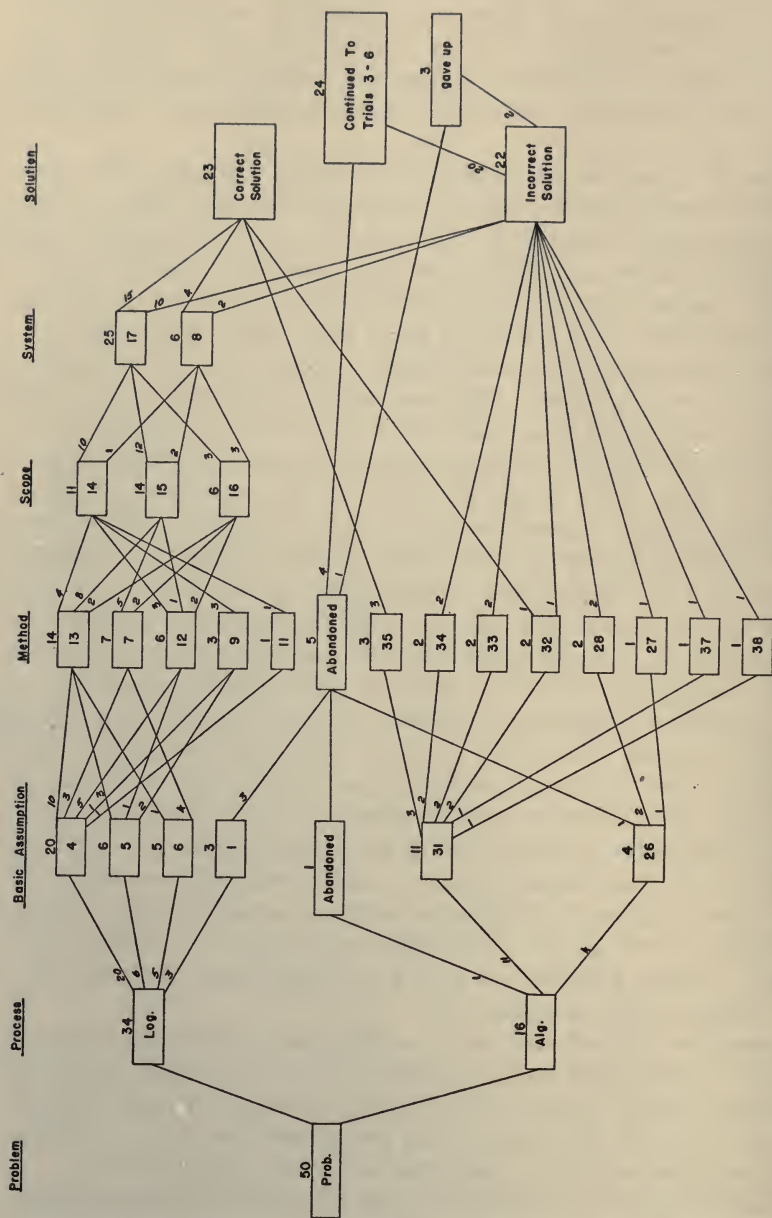


Fig. 11. Patterns of problem-solving thinking of 50 college students on the donkey-mule problem, second trial.
Number of subjects above rectangles. Operations within rectangles.

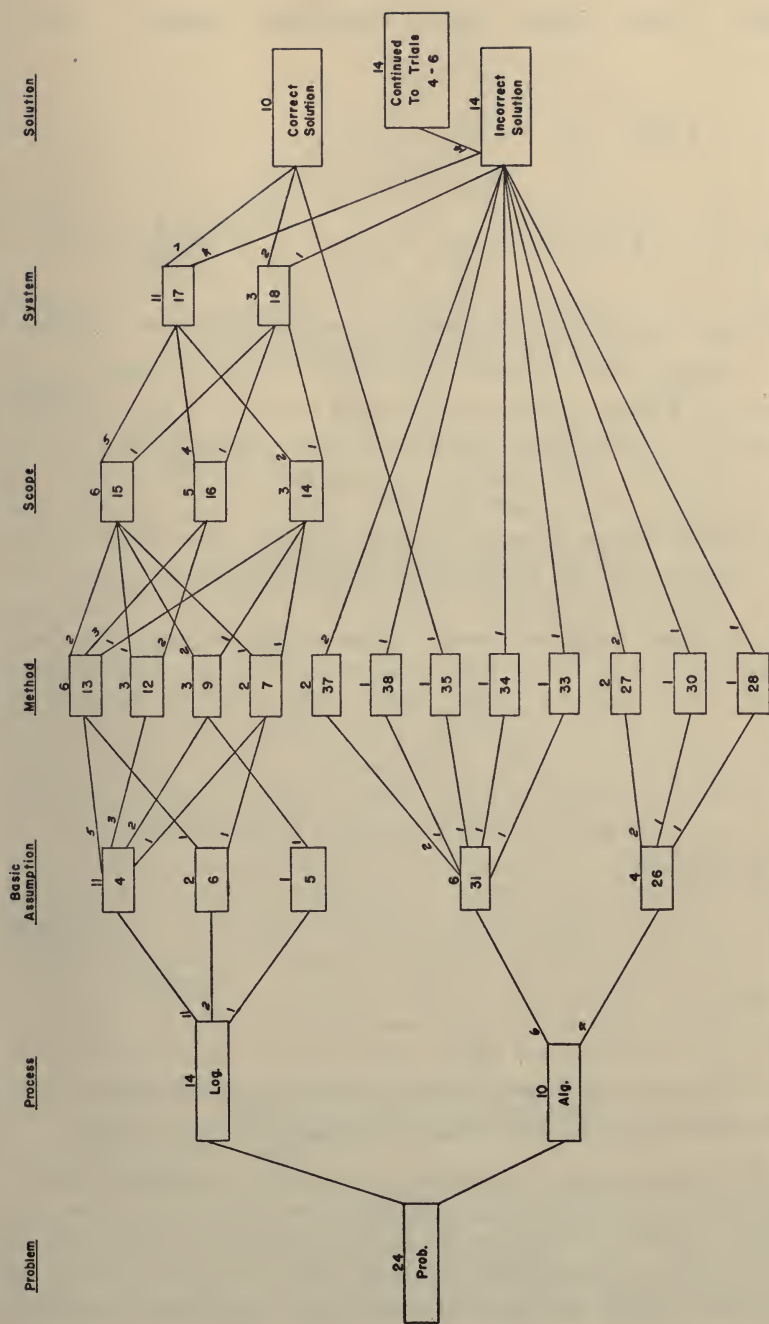


Fig. 12. Patterns of problem-solving thinking of 24 college students on the donkey-mule problem, third trial. Number of subjects above rectangles. Operations within rectangles.

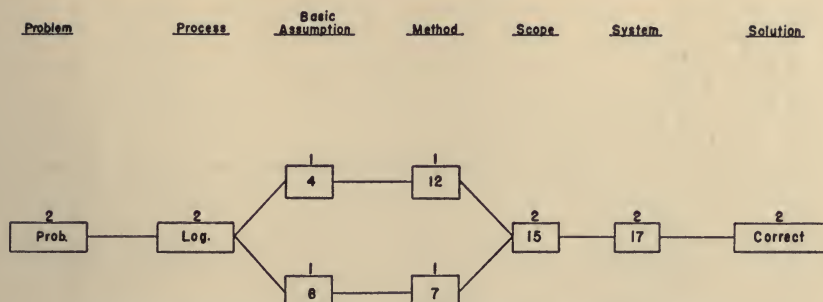


Fig. 15. Patterns of problem-solving thinking of 2 college students on the donkey-mule problem, sixth trial.
Number of subjects above rectangles. Operations within rectangles.

by numbers 1 to 6. In the first trial this subject used the process of logical reasoning and chose successively Cards 4-12-14-18. Unable to get a solution, she continued to a second trial, using the logical process and choosing Cards 4-9-14-17. She gave the incorrect answer 2 and 3 gallons and then proceeded to trial 3. In this trial she tried the algebraic process, using equations with one unknown (Card 26) and then selecting Card 27. This led to an incorrect answer; so she began trial 4, again using algebra but selecting Cards 31 and 34, and again got a wrong answer. On trial 5 she returned to logical reasoning and selected Cards 5-9-15-18 but stopped before getting an answer. On the sixth trial, using logical reasoning, she chose Cards 6-7-15-17 and gave the correct answer, namely, 5 gallons and 7 gallons. Her shift from logical reasoning to algebra and then back to logical reasoning shows her lack of a consistent pattern of thinking in attacking problems such as this.

The steps for successive trials will be shown in table 19 rather than in a chart showing the variations in individual patterns of thinking of each subject. Subject 219, whose record was shown in fig. 16, appears as serial number 60 in table 19, and the pattern shown in the chart may be verified from the card numbers for the successive trials given in the table. Although table 18 showed the total number of subjects following each pattern for each trial, it did not indicate the successive patterns followed by individual subjects from trial to trial. However, a tracing of the complete sequence of thinking for all subjects and for all trials is given in table 19.

Of the 61 subjects shown in table 19, there is one group of 3 subjects (serial nos. 4, 5, and 6) that followed identical patterns of solution. There are four groups of 2 subjects each (serial nos. 1 and 2, 7 and 8, 22 and 23, and 29 and 30) that followed identical patterns. Of the remaining 50 subjects, each followed a different pattern of thinking

TABLE 19
SEQUENCE OF OPERATIONS FOR 61 COLLEGE STUDENTS IN SOLVING THE DONKEY-MULE PROBLEM

Serial no.	Subject no.	Card numbers of operations					
		Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Trial 6
1	231	L-4-12-15-17					
2	275	L-4-12-15-17					
3	213	L-4-13-15-17					
4	217	L-4-13-16-17					
5	253	L-4-13-16-17					
6	262	L-4-13-16-17					
7	216	L-6-7-15-17					
8	288	L-6-7-15-17					
9	264	L-6-7-15-18					
10	229	L-6-13-15-17					
11	208	A-31-35					
12	276	L-1	L-4-12-16-17				
13	289	L-1	L-4-13-15-17				
14	282	L-1	L-6-7-15-17				
15	214	L-2-10-14-18					
16	243	L-3-8-14-17					
17	271	L-3-13-15-17					
18	232	L-4-7-14-18					
19	223	L-4-9-14-17					
20	283	L-4-12-14-17					
21	234	L-4-12-14-17					
22	287	L-4-13-14-17					
23	256	L-4-13-14-17					
24	226	L-4-13-14-17					
25	201	L-4-13-14-18					
26	245	L-5-9-15-17					
27	239	L-5-9-15-17					
28	227	L-6	L-4-7-16-18				
29	260	L-6-12-14-17	L-5-13-15-18				
30	250	A-26	A-31-35				
31	237	A-26	A-31-35				
32	209	A-26-29	L-5-13-16-17				
33	236	A-31	A-26-27				
34	257	A-31	L-4-13-15-17				
35	248	A-31	L-4-7-15-17				
36	235	A-31-32	L-5-13-15-17				
37	251	A-31-34	A-31-35				
			A-31-32				

TABLE 19—Continued

Serial no.	Subject no.	Card numbers of operations					
		Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Trial 6
38	286.....	L-1	L-4-11-14-17	L-4-12-16-17			
39	273.....	L-1	L-4-13-14-17	L-4-13-15-17			
40	221.....	L-1	L-5-9-14-17	L-4-13-15-17			
41	203.....	L-3-8-14-17	L-4-12-14-17	L-4-12-16-17			
42	241.....	L-4-12-14-18	A	L-4-13-16-18			
43	270.....	L-5-9-14-17	L-4-13-14-17	L-4-12-15-18			
44	246.....	L-5-9-15-18	L-4-13-14-17	L-4-13-16-17			
45	225.....	L-6-7-14-17	A-31-33	A-31-35			
46	280.....	A-26-27	L-4-13-14-17	L-4-13-16-17			
47	206.....	A-26-27	A-31-33	L-4-7-15-17			
48	207.....	L-1	L-4-12-14-18	L-4-9-15-17	L-4-13-16-17		
49	240.....	L-4-11-14-18	A-26-28	A-26-27	L-4-7-16-17		
50	211.....	L-4-12-14-17	A-31-34	A-31-33	A-31-35		
51	249.....	L-5	A-26-28	A-26-30	L-6-7-15-17		
52	244.....	A-26-28	L-1	L-5-9-15-17	L-6-7-15-17		
53	212.....	A-31-33	A-31-32	A-31-37	L-6-7-15-17		
54	247.....	A-31-34	A-31-37	A-31-38	A-31-35		
55	228.....	A-31-34	L-4-12-14-17	A-31-37	L-6-7-15-17		
56	233.....	A-31-36	L-5-9-14-17	L-4-9-14-18	A-31-35		
57	279.....	A-31-37	A-31-38	L-6-7-14-17	L-2-10-15-17		
58	290.....	L-4-13-14-18	L-1	L-6-13-14-17	A-31-35		
59	255.....	A-31-37	A-31-34	L-6-13-14-17	A	L-4-13-15-17	
60	219.....	L-4-12-14-18	L-4-9-14-17	A-26-27	L-4-11-14-17	L-4-11-15-17	
61	230.....	A-26-29	A-26	A-31-34	A-31-34	L-5-9-15-18	
					L-1	L-4-12-14-17	L-6-7-15-17 L-4-12-15-17

L indicates logical method; A indicates algebraic method.

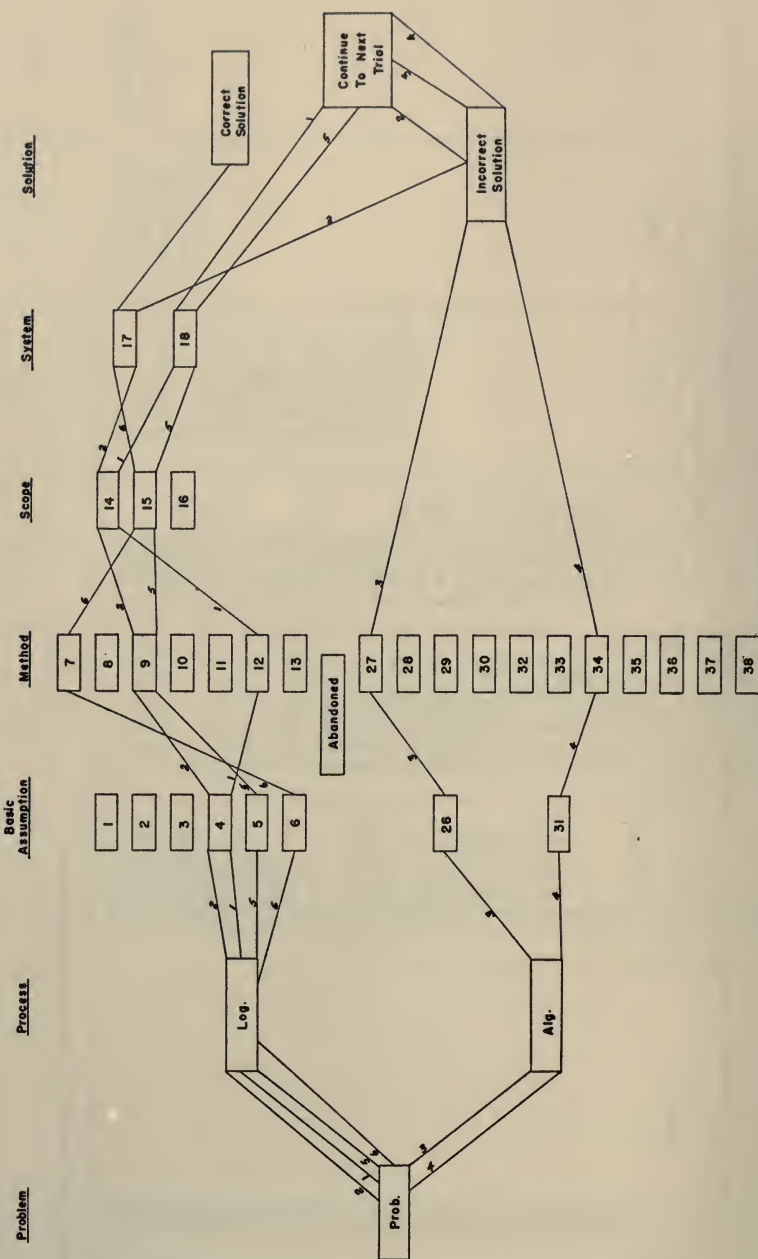


Fig. 16. Patterns of Subject 219 for 6 successive trials on the donkey-mule problem.

TABLE 20
CARDS SELECTED IN SUCCESSFUL AND FAILING TRIALS
(LOGICAL REASONING)

Card numbers	Number of times each process card was selected	
	Correct solution	Failures
Assumption 1.....	0	11
2.....	0	2
3.....	0	3
4.....	30	26
5.....	4	9
6.....	13	5
Total.....	47	56
Method 7.....	15	4
8.....	0	2
9.....	0	12
10.....	0	2
11.....	1	3
12.....	9	10
13.....	22	10
Total.....	47	43
Scope 14.....	0	34
15.....	32	8
16.....	15	1
Total.....	47	43
System 17.....	40	31
18.....	7	12
Total.....	47	43

Because 13 subjects abandoned the trial after selecting the assumption, the totals are 43 rather than 56 in sections on method, scope, and system.

in solving the problem. An examination of the card sequences, subject by subject, gives an impression of trial and error rather than of a systematic and coherent method of attacking the problem. The evidence from this group of college subjects indicates variation rather than uniformity as the major characteristic of the problem-solving process. This fact is even more impressive when one recalls that the 38 process cards had been condensed from a total of 114 cards at the end of the second stage of the study carried on with high school pupils. Even with

the amount of generalizing necessary to reduce the variations from 114 to 38 cards, there is still very little agreement in the sequence of thinking of the subjects studied.

Although the sequence of thinking shows little similarity from subject to subject, there is a marked difference in the operations employed in the trials that led to a successful solution as compared with the trials that resulted in failure. An analysis of the data for the trials that used logical reasoning is given in table 20. The table should be read as follows: In the 47 trials that resulted in a correct solution, no subject began by selecting Card 1, whereas in the 56 trials that resulted

TABLE 21
NUMBER OF SUBJECTS FOLLOWING DIFFERENT PATTERNS
IN FINAL SUCCESSFUL TRIAL

No.	Patterns on successful trial	Number of subjects
1	4-13-15-17.....	9
2	6-7-15-17.....	9
3	4-13-16-17.....	6
4	4-12-15-17.....	4
5	4-12-16-17.....	3
6	4-7-15-17.....	2
7	4-13-16-18.....	2
8	6-7-15-18.....	2
9	6-13-15-17.....	2
10	4-7-16-18.....	1
11	5-12-16-17.....	1
12	4-7-16-17.....	1
13	4-11-15-17.....	1
14	4-12-15-18.....	1
15	5-13-15-17.....	1
16	5-13-15-18.....	1
17	5-13-16-17.....	1
	Total.....	47

in failure, 11 chose Card 1. None of those who succeeded began with an assumption given on Cards 1, 2, or 3, but 16 of those who failed selected one of these wrong-assumption cards. The data in table 20 show also that 16 of the subjects who failed selected Method Cards 8, 9, or 10, which were selected by none of the subjects that succeeded. In thirty-four of the failing trials, the subjects selected Scope Card 14, which was chosen by none on the successful trials. Forty of the 47

who made successful trials selected System Card 17. A systematic procedure does not correct a mistaken assumption or a wrong method, but with correct assumptions and methods it gives a marked advantage over random procedure.

One further analysis will be made in the search for common patterns of problem solving. Table 21 shows the sequence of operations in the final successful trial of the 47 subjects who succeeded through the use of logical analysis. There were two groups of 9 subjects each, one group of 6 subjects, and one group of 4 subjects who followed identical patterns. The remaining 19 subjects followed 13 different patterns. Although the table shows a certain amount of agreement, the fact remains that, even in the trials leading to a correct solution, many different patterns were followed. The evidence gives no support to the notion that problem solving must follow the neat, precise recipes that are so frequently encountered in textbooks on methodology.

CHAPTER VI. SUMMARY AND CONCLUSIONS

The study reported in the foregoing chapters falls into three general divisions. Chapter ii dealt with certain preliminary procedures in problem solving; chapter iii presented data on the discovery of generalizations, and chapters iv and v reported an analysis of procedures in solving a single problem. The major concern throughout the study was with the processes involved in the solving of problems and in the sequence of the operations employed.

PRELIMINARY FACTORS

When a problem is presented in writing, the initial step is to read it with sufficient care to recognize the general logical conditions set forth and to identify and separate the relevant facts given from any irrelevant ones that are included. If the problem solver correctly understands logical conditions, he is then in position to make the necessary assumptions and to decide on the general method that will lead to a solution of the problem; and if he has accurately separated the relevant facts from the irrelevant ones, he is able to apply the method to proper specifics. A correct reading of a problem is an essential condition for its solution.

The data secured from Test 1 and presented in tables 2 to 5 showed that failure to solve a problem is very often due to inability to separate relevant from irrelevant facts or to recognize a need for further facts not given in the statement of the problem. Tables 4 and 5 showed the wide variation in answers that results from failure to identify relevant,

irrelevant, and needed facts. Many of the failures in solving the problems can be traced to inadequate reading of the conditions of the problem.

The second preliminary factor studied was the estimation of an answer before the subject proceeded to find the exact answer. The purpose here was to give meaning and reality to the operations employed and to set up a guard against unreasonable results. Test 2 dealt with this factor, and the findings were presented in tables 6 and 7. The data indicated that a large number of subjects were unable to make a reasonable estimate of the answer and that many used methods of estimating that were crude and time-consuming. Although most textbooks in arithmetic give some space to the estimation of answers, many of the high school and college students in this study showed little understanding of techniques for doing this. The analysis of methods of approximating, as indicated in table 4, showed the wide variety of procedures that were followed.

The third preliminary test undertook to find whether, in three-step problems, there was uniformity in the sequence of operations. The data were presented in figures 1 to 8 and in table 9. The problems used were purposely simple, and the objective was in part to find whether variation in sequence could be diagrammed effectively, for use in a later part of the study. What was more important, however, was to find whether there were a few commonly followed patterns or whether there was little agreement in sequence. Even for such simple problems as were used, the evidence from the charts and table 9 showed considerable variability in sequence of operations. The logic implied in the verbal statement of the problem had a somewhat greater effect on the sequence of operations than the order of presentation had. Judging from the percentage of correct answers, some sequences are more effective than others. Here, again, the problem solver must, as a preliminary step, decide what he will do first, and then next, and next.

The fourth factor in the preliminary group of tests had to do with the use of numbers and letter symbols, as indicated by the data given in tables 10 and 11. Problems expressed in letter symbols gave much more difficulty than similar problems expressed in numbers. Many of the subjects omitted the letter-symbol problems, evidently because they did not know how to use such symbols. All the subjects were high school or college students, many with a background of algebra who presumably had been taught the value of the generalized letter symbols in dealing with problems. Yet there was a marked feeling of insecurity among them when they were given problems expressed in abstract symbols.

The data from the four tests in the first section of this study indicated that some deficiencies in problem solving reside in certain conditions preliminary to the central logical thinking employed in solving the problems. Better training (*a*) in reading problems in order to sense relevant, irrelevant, or needed data, (*b*) in estimating reasonable answers, (*c*) in following a superior sequence of operations, and (*d*) in using abstract symbols might be expected to contribute to competence in problem solving.

DISCOVERING GENERALIZATIONS

The second part of the study, reported in chapter iii, dealt with the process of solving a problem by discovering a generalization that would cover the conditions of the problem. The problem was to discover a generalization by means of which the sum of a column of even numbers could be obtained more quickly than by the usual method of adding. A second problem involved getting the sum of a similar column of odd numbers beginning with 1. The data for this part of the study were presented in tables 13 to 16.

A time record was obtained for the examples in Test 5. The hypothesis for this part of the study was that the discovery of a suitable generalization would be followed by a sharp reduction in the time needed for adding a column of digits. The sample of data given in table 13 showed no consistent pattern in support of the hypothesis, for reasons explained in chapter iii. However, it seems reasonable that a change in design of the experiment, using columns of equal length and difficulty, might be successful in showing just where, in the sequence of thinking, the generalization was discovered.

The second question studied in this part of the research concerned the level or quality of the generalizations that were given by the subjects. The data consisted in voice recordings and notes made by the experimenter as the subject tried to discover a generalization. Seven types of generalizations were given by the subjects for Problem 1, and nine types for Problem 2. The frequency of the types was shown in tables 14 and 15. Three generalizations, each correct, were stated by 37 subjects, whereas 32 subjects arrived at one of 4 incorrect generalizations. For Problem 2 (table 15), 23 subjects gave a correct generalization; the remaining 45 gave 7 different incorrect statements. The generalizations represented different levels of value, some methods at the lower extreme being correct but so cumbersome that they took little less time than adding.

The principal value from the study of generalizations, however, is not found in the statistical tables but rather in the protocols showing the actual process of reaching a satisfactory generalization. The process of arriving at a generalization is far from simple, and the confusion and frustration in thinking exhibited by these 69 subjects, all college students, evidenced lack of familiarity with the nature of formulating generalizations.

Problem 3 of Test 5 was used to test the transfer of a generalization. Only 13 of the 23 subjects who made a correct generalization for finding the sum of a column of odd numbers beginning with 1 were able to transfer it to the pyramid of x marks in Test 3. Apparently those who failed to transfer the generalization understood it only at a lower level, applicable only where columns of numbers are concerned. Thus the full import of the possibilities of using generalizations escaped them.

PATTERNS OF PROBLEM SOLVING

Chapters iv and v reported the extent to which there were common patterns of thinking in the process of solving problems. The particular feature of this part of the study was the attempt to construct and then apply an instrument that would permit objective comparisons of the thinking process. The construction of the instrument was described in chapter iv, and the results of its use in chapter v. The data were presented in figures 10 to 16 and in tables 17 to 21.

The particular dilemma in designing this part of the study was to leave the process of thinking as unstructured as possible but at the same time describe it objectively in terms of a set of categories that expressed the different steps in process. This necessarily required some compromise, but one essential criterion was adhered to, namely, that the categories be derived from the records of thinking of the subjects rather than formulated in advance by the experimenter.

The major finding of this part of the study was that, although some agreement in patterns of thinking was revealed, the most striking characteristic was variety, rather than similarity, in sequence of thinking. Even when comparisons were limited to the trials that produced a successful solution of the problem (table 21), in no case were as many as 20 per cent of the group represented by any one pattern of thinking. Among the 47 trials resulting in a correct solution, there were 17 different patterns. If the complete sequence of thinking, including both successful and unsuccessful trials, is considered (table 18), there were 51 different patterns in a total of 155 trials. One is forced to the con-

clusion that either (a) great variety in the process of problem solving is normal and to be accepted or (b) schools have been ineffective in teaching a technique of problem solving, or, as often expressed, "teaching students how to think."

The data presented in figures 10 to 16 show strikingly the agreements and disagreements in patterns followed. Figure 16, which showed the six successive trials of one subject, does not picture a process in which an initial trial is corrected consistently trial by trial until a solution is reached. Rather it pictures a variation in sequence with little coherence evident. The data in table 19, which gave the detailed progress of each subject from start to finish, again showed that problem solving proceeds in ways that indicate a lack of a technique of thinking. The oscillating from algebra to logic, or vice versa, gives an indication of bafflement rather than a feeling of security in the use of either logical or algebraic procedures.

IMPLICATIONS FOR SCHOOLS

1. The data in the study indicate the need for increased ability to read problems. In the solution of the "donkey-mule" problem there was repeated indication of wrong assumptions due to mistakes in reading, and numerous rereadings were required for an understanding of the correct conditions of the problem. Likewise, in Test 1 the basic difficulty was failure to read critically in order to separate relevant and irrelevant facts. In some schools explicit attention and much practice is given to the reading and interpretation of problems; yet in most schools this type of experience is left to arithmetic and science classes in which many teachers do not consider teaching students to read as part of their assignment. The teaching of critical reading must be extended beyond the elementary school if it is to function in areas where problem solving is most commonly found.

2. The evidence in the section of the study that dealt with discovering generalizations indicated lack of maturity in this type of thinking. Even the verbalization of generalizations by subjects who apparently had the correct idea was accomplished with great difficulty. Many of the generalizations were expressed in painfully awkward terms. One infers that these college students were more accustomed to learning generalizations when presented in finished form than they were to discovering them. Economy of time may necessitate the teaching of many principles expressed in abstract form, but some amount of attention to finding generalizations is necessary if students are to cope with problems for which discovery of a generalization is a cue to the solution.

3. The study revealed a variety of patterns of problem solving that eventuate in correct answers. Just any pattern is not necessarily as valid as any other pattern, but certainly there is more than one path to a correct solution. One may be somewhat wary of teaching which talks of "the way to solve problems." There have been many examples of methods which have stressed such a single path for problem solving; yet in many cases the result has been a formalizing of thinking which made solving a problem much like following a recipe in the kitchen. The variety of patterns found in the present study would at least warrant a considerable degree of freedom in solving a problem as long as the thinking is genuine, at least until superior methods are discovered and made habitual.

4. The major purpose of the study may be realized if it induces teachers to study the patterns of problem solving employed by their students. The study has indicated that it is possible to express patterns of problem solving in objective form by recording what students do as they solve a problem. In very simple problems, such as those used in chapter ii, Test 3, a chart of the sequence of operations may be made from the written work of a group. If such a chart is shown to the group, and if the question is asked why some students think in one sequence and some in another, the students become aware of what is meant by a pattern of thinking and the way is cleared for a study of their own methods of solving problems. In the solution of more complicated problems, such as the one used in chapter v, a student's pattern of solution may be studied by first observing his reading of the problem and the assumptions he makes in beginning to solve it. Then, in sequence, one may observe which of the possible methods of solution he chooses, in what ways he may limit the scope of his method, the extent to which his procedure is systematic or random, and whether he verifies the solution he obtains. The essential point is that the teacher try to understand what the student thinks *as he solves* the problem and that any method by which evidence on the process of thinking is obtained is helpful.

If, in general, teachers would shift their major concern from the answers obtained to the methods of thinking by which answers are obtained, there is a possibility that more effective thinking would result.

5. It has become customary for schools to express the major objective of education as "teaching students how to think." From all that has been written about this objective one might infer that students are now learning a certain coherent logical procedure for solving problems and that evidence of such a superior process of reasoning would be revealed

in the patterns of thinking that are employed. If schools are successful, this use of logical thinking should be one of the most obvious characteristics of an educated person. The data from the present study give little evidence that the schools are achieving any marked success in reaching this objective. Rather, the thinking of the subjects in this study indicates an absence of a generalized mode of problem solving.

The teaching of courses in formal logic was sharply reduced many years ago on the grounds that better methods of thinking would result from teaching in situations that were more functional and lifelike. It is difficult to find objective evidence to justify the change. Neither the earlier courses in formal logic nor the more recent functional teaching has produced outcomes that are satisfying. The present study simply adds to the evidence that "teaching students to think" is more of a stereotype from college catalogues than a realized outcome of experience in the schools. There are numerous hypotheses for improving the situation. Some persons urge a newer type of mathematical logic in the college; some a return to systematic teaching of grammar in the elementary school; some an attempt to suffuse meaning into algebra in the high school. And there are others. Whatever experimental changes are tried, this study would suggest that some procedure for revealing actual patterns of thinking might supply objective evidence for judging the value of the change. The purpose of this study has been to develop a possible method of identifying patterns of thinking.

This monograph has dealt with a problem that is basic in educational theory. The dilemma is whether the school is to "train" students in specific ways of thinking by means that stereotype the process and limit the outcome to the specific areas covered, or whether the school can help the student to learn to think in ways that provide versatility and effectiveness in dealing with unpredictable situations. Methods of teaching students how to think must not only avoid the danger of stereotyping the process of thinking but also the hazard of developing such variability in thought patterns that effective and ineffective processes of thinking are not differentiated. The authors of this monograph believe that knowledge of the detailed operations employed in solving a problem will contribute to the finding of superior methods of teaching students how to think.

APPENDIX

TESTS USED IN THE STUDY

Name:	Last	2	First	3	Middle	4	5	Class	Age	6	Sex	No.

PROBLEM-SOLVING EXERCISES

The purpose of the exercises in the following pages is to indicate the mental processes employed in solving problems as well as to measure the ability to get correct answers.

Tests 1 to 4: The directions for each of these tests is given at the top of their respective pages. Read the directions carefully. Then glance at the blackboard and write in the space marked *Time* the time when the test is finished.

When you have finished test 4, turn in your test booklet. Do not look ahead to the following material.

Tests 5 and 6: These are individual tests. Directions for these will be given at the time the test is taken.

TEST 1

SELECTING DATA

Each of the problems below gives some data that are necessary for solving them. In some cases additional data are given that are *irrelevant* to the problem. In some cases *insufficient* data are given and additional facts must be supplied before the problem can be solved. Indicate the relevant facts, irrelevant facts, and additional facts needed as is done in the sample below. Do not attempt to solve the problems.

Sample: A man bought 7 stamps on May 29 and gave the clerk a half dollar. How much change should he receive?

<i>Relevant facts</i>	<i>Irrelevant facts</i>	<i>Facts needed</i>
7 stamps Half dollar	May 29	Denomination of stamps

1. It takes 8 men 2 days to dig a 160-foot drain. Their wages are \$1.50 an hour. How many men are needed to dig the drain in half a day?

<i>Relevant facts</i>	<i>Irrelevant facts</i>	<i>Facts needed</i>

2. On January 14, Mr. Smith and his son who was 17 years old drove to market with some sheep which they sold for 35 cents a pound. If the average weight of the sheep was 60 pounds, what was the amount of money received?

<i>Relevant facts</i>	<i>Irrelevant facts</i>	<i>Facts needed</i>

3. A ship has provisions to last her crew for 6 months. The crew uses 1200 gallons of water, 24 sacks of potatoes, and 400 pounds of meat every 5 days. How long would these provisions last a crew of 800 men?

<i>Relevant facts</i>	<i>Irrelevant facts</i>	<i>Facts needed</i>

4. A tank with a capacity of 150 gallons is used to irrigate 2 acres of land. After $1\frac{1}{2}$ acres of land were irrigated, the tank was $\frac{1}{2}$ full. How many minutes will it take to fill the tank at the rate of $7\frac{1}{2}$ gallons per minute?

<i>Relevant facts</i>	<i>Irrelevant facts</i>	<i>Facts needed</i>

TEST 2

ESTIMATING ANSWERS

Without solving for the exact answer, write in the box at the right what you think would be an approximate answer. Then explain in the space provided how you made your estimate.

1. A machine that sold regularly at \$985.00 was priced at a 10% discount. What was the discounted price?

Explain _____

Approx. Ans.

2. At an auction sale, 98 books sold for a total of \$497.50. What was the average amount received per book?

Explain _____

Approx. Ans.

3. The attendance at five football games was as follows: (1) 20,127; (2) 17,013; (3) 23,106; (4) 19,775; (5) 19,876. What was the total attendance?

Explain _____

Approx. Ans.

4. In a community chest drive, three high schools collected amounts as follows: \$994.70; \$789.50; and \$1,236.00. What was the average amount collected by the three schools?

Explain _____

Approx. Ans.

TEST 3

PROBLEM SOLVING SEQUENCE

Solve the problems below. Do all of your work in the spaces provided, writing what you do first in the first space, what you do next in the second space, and so on. Write your final answers in the boxes. Check (verify) your answer in the space at the bottom.

1. If 15 hens each lay 5 eggs per week for 4 weeks, what is the total value of the eggs at 6 cents each?	2. At 60 cents per hour, find the total earnings of 12 boys if they each work 8 hours per day for 5 days.	3. If all seats are sold, find the total daily receipts for a theater that contains 20 rows of seats with 24 seats per row, charges 50 cents admission, and has 3 shows per day.
1.	1.	1.
2.	2.	2.
3.	3.	3.
4.	4.	4.
Answer	Answer	Answer
<i>Check</i>	<i>Check</i>	<i>Check</i>

TEST 4

SYMBOLIC THINKING

Write the correct answer in the box following each problem.

Sample: How much is \$5 and \$4?

- (1) \$6 (2) \$9 (3) \$10 (4) \$7 (5) $\frac{\$4}{5}$

\$9.00

1. How much is r and t dollars?

- (1) $r + t$ (2) $t - r$ (3) $r \times t$ (4) $r \div t$ (5) $\frac{r}{t}$

2. Mr. Jones received a 5% discount on a car originally priced at \$2100. How much was the discount?

- (1) \$105 (2) \$90 (3) \$1050 (4) \$10.50 (5) \$205

3. By mistake, Johnny multiplied a fraction by 3 instead of dividing by 3. He gave the answer as $2/3$. What should the correct answer be?

- (1) $\frac{2}{9}$ (2) $\frac{2}{27}$ (3) 3 (4) $\frac{6}{9}$ (5) 6

4. A boy, by mistake, multiplied a fraction by m instead of dividing by m . He gave as his answer $\frac{x}{m}$. What was the correct answer?

- (1) $\frac{x}{m} + (m \times m)$ (2) $\frac{x}{m} \times m$ (3) $\frac{x}{m} \times \frac{1}{m} \times \frac{1}{m}$
 (4) $\frac{x}{m} \times \frac{m}{1} \times \frac{m}{1}$ (5) $\frac{x}{m^3}$

5. The living room rug is 6 feet by 12 feet, and the hall rug is 3 feet by 4 feet. How many times larger is the living room rug than the hall rug?

- (1) $2\frac{4}{7}$ (2) 90 (3) $1\frac{1}{4}$ (4) 6 (5) $\frac{2}{3}$

6. Mr. Jones sold a car for n dollars and received $p\%$ commission on the sale price. How much was his commission?

- (1) $n + p$ (2) $n - p$ (3) $p \times n$ (4) $\frac{n}{p}$ (5) $n \times p + n$

7. The vegetable garden is r feet long and y feet wide. The flower garden is n feet long and k feet wide. How many times larger is the vegetable garden than the flower garden?

- (1) $\frac{r+y}{n+k}$ (2) $(r \times y) + (n \times k)$ (3) $\frac{r}{n} + \frac{y}{k}$ (4) $\frac{r \times y}{n \times k}$ (5) $\frac{r}{y} + \frac{n}{k}$

TEST 5

DISCOVERING GENERALIZATIONS

General directions: In the three sets of problems below try to discover some method or some general rule that will enable you to get the answers more rapidly than by ordinary computation. When you have finished each set of exercises, write in the space provided the generalization or rule which you have discovered.

1. Can you find the sum of a *series* of numbers (such as 14, 15, 16, 17, 18, 19, 20) more quickly than by adding the numbers in sequence? Find the sum of exercises *a* to *k* and write the answer under each column. When you have finished Ex. *k*, write the method that you have discovered.

<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	<i>i</i>	<i>j</i>	<i>k</i>
1	4	3	2	1	22	4	96	997	44	12
2	5	4	3	2	23	5	97	998	45	13
3	6	5	4	3	24	6	98	999	46	14
		6	5	4	25	7	99	1000	47	15
		7	6	5	26	8	100	1001	48	16
			7	6	27	9	101	1002	49	17
			8	7	28	10	102	1003	50	18
				8		11	103	1004	51	19
				9		12	104		52	20
						13			53	21
						14			54	
						15			55	
						16			56	

Generalization _____

2. Can you find the sum of a series of odd numbers (such as 1, 3, 5, 7, 9, 11) without adding? Following are some series of odd numbers. Write the sum of each series as quickly as possible. Then write any method or generalization that you can discover to enable you to find the sums more rapidly. The numbers in parentheses at the top of the columns indicate the number of odd numbers in the series.

(a) (2)	(b) (4)	(c) (5)	(d) (3)	(e) (7)	(f) (8)
1	1	1	1	1	1
3	3	3	3	3	3
—	5	5	5	5	5
	7	7	—	7	7
	—	9		9	9
		—		11	11
				13	13
				—	15
					—
(g) (12)	(h) (15)	(i) (17)	(j) (14)	(k) (13)	(l) (16)
1	1	1	1	1	1
3	3	3	3	3	3
5	5	5	5	5	5
7	7	7	7	7	7
9	9	9	9	9	9
11	11	11	11	11	11
13	13	13	13	13	13
15	15	15	15	15	15
17	17	17	17	17	17
19	19	19	19	19	19
21	21	21	21	21	21
23	23	23	23	23	23
—	25	25	25	25	25
	27	27	27	—	27
	29	29	—		29
	—	31			31
		33			—
		—			

3. How many x marks are in the pyramid below? There are 9 in the center column.

```

      |x|
    x|x|x
  x|x|x|x|x
x|x|x|x|x|x|x
x|x|x|x|x|x|x|x
x|x|x|x|x|x|x|x|x
x|x|x|x|x|x|x|x|x|x
x|x|x|x|x|x|x|x|x|x|x
x|x|x|x|x|x|x|x|x|x|x|x

```

Answer

Generalization _____

TEST 6

PROBLEM SOLVING PROCESSES

Directions: Make whatever notations or computations you wish in the spaces below. "Think aloud" as you work in order to give as full a record as possible of what you do while you are solving the problem.

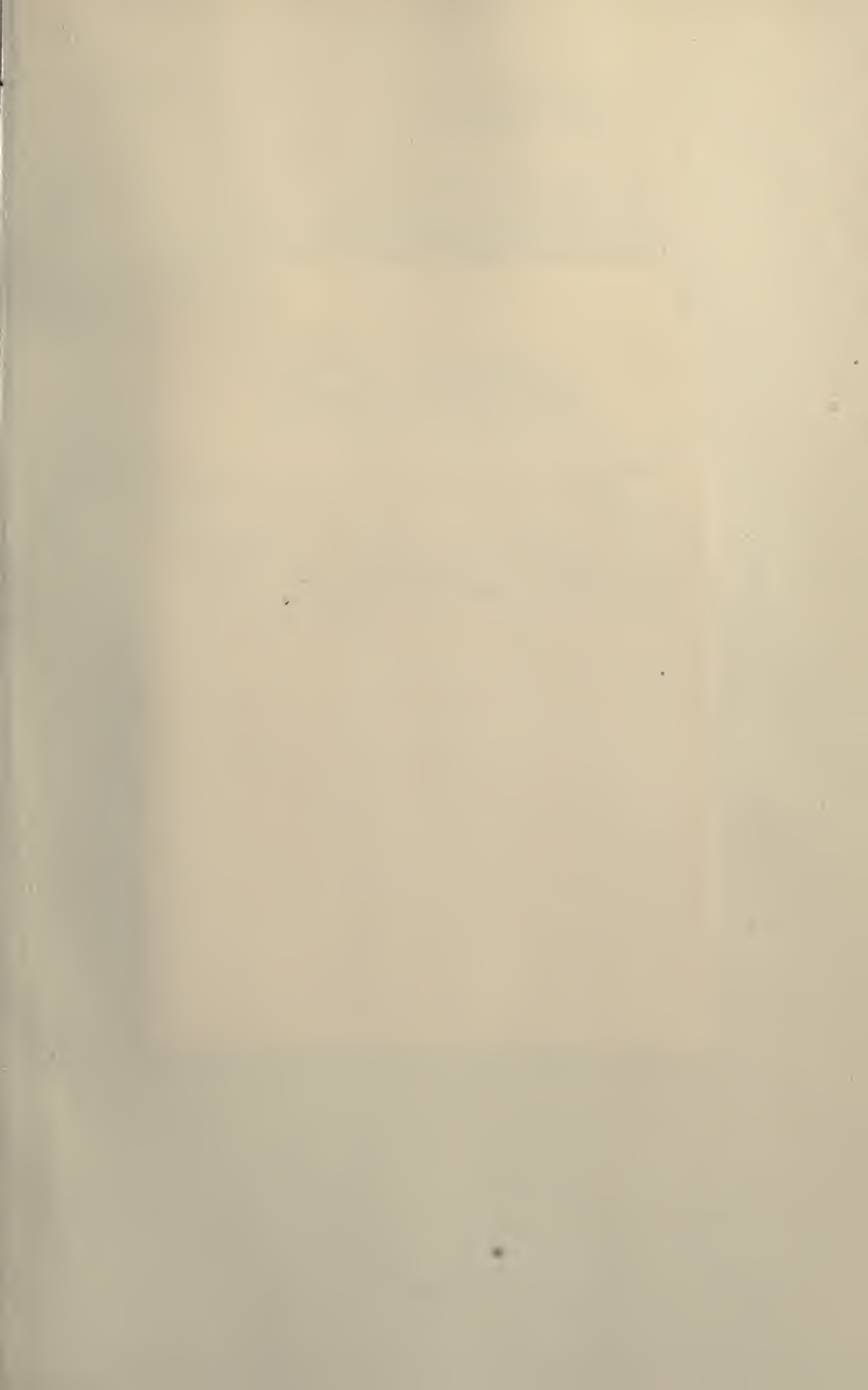
Problem: A donkey and a mule were laden with wine. If 1 gallon of wine was taken from the donkey and placed on the back of the mule, the mule would have double the load of the donkey. But if 1 gallon was taken from the mule and placed on the donkey, the two animals would have equal burdens. How many gallons of wine did each animal carry at first?

1.

2.

3.

4.



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